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Abstract. We introduce a general class of random spin systems which are symmetric under local gauge transformations. Our model is a generalization of the usual Ising spin glass and includes the Z_q, XY, and $SU(2)$ gauge glasses. For this general class of systems, the internal energy and an upper bound on the specific heat are calculated explicitly in any dimensions on a special line in the phase diagram. Although the line intersects a phase boundary at a multicritical point, the internal energy and the bound on the specific heat are found to be written in terms of a simple function. We also show that the boundary between the ferromagnetic and non-ferromagnetic phases is parallel to the temperature axis in the low-temperature region of the phase diagram. This means the absence of re-entrant transitions. All these properties are derived by simple applications of gauge transformations of spin and randomness degrees of freedom.

1. Introduction

The $\pm J$ Ising spin system is a typical model of spin glasses. When the bond distribution is symmetric (that is, the concentration p of ferromagnetic bonds is 1/2), the existence of the spin glass phase at finite temperatures has been discussed extensively for finite-dimensional systems. The lower critical dimension is now believed to be between two and three [1–3]. In the asymmetric case ($p > 1/2$), the phase diagram in finite dimensions with paramagnetic, ferromagnetic and spin glass phases has been determined by numerical calculations [4–8] and renormalization group arguments [9].

For the problem of the precise topology of the phase diagram of the $\pm J$ Ising model, there is a powerful technique to derive a variety of rigorous and exact results which set strong constraints on the topology of the phase diagram [10]. This technique, the method of gauge transformation, shows that the internal energy and an upper bound on the specific heat can be calculated exactly as non-singular functions of p or T on a special line defined by $\exp(-2J/k_BT) = (1-p)/p$ in the p–T phase diagram. It has also been established using similar ideas that there is no re-entrant transition from the ferromagnetic phase to a non-ferromagnetic phase as the temperature is lowered at a fixed concentration of ferromagnetic interactions [11, 12]. These results also hold for Ising spin glasses with Gaussian random exchange interactions.

In the case of the classical XY model, the lower critical dimension of the usual spin glass ordering is believed to be four or larger in the $\pm J$-type random model [13–18], although the possibility of chiral spin glass freezing in three dimensions has been pointed out [16, 17]. Recently another type of a random XY model, the XY gauge glass, has attracted much attention [19–29]. It is suggested that the lower critical dimension of the XY gauge glass lies between two and three [24–29]. Investigation of the phase diagram of these random XY models is still at a primitive stage compared with the case of the Ising model. There is

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controversy about the existence of re-entrance in the weakly random region of the XY gauge glass in two dimensions. Some real-space renormalization group calculations suggest re-entrance [19–21] which Monte Carlo simulations and Migdal–Kadanoff-type renormalization group transformations have failed to confirm [22, 23, 29].

It has been known for many years that the method of gauge transformation is applicable to a wider class of models such as the XY gauge glass model [10], the random Potts model [30] and the Ashkin–Teller model [31] to obtain the internal energy, specific heat and correlation inequalities. In the present article, we generalize this idea and derive conditions on a generic Hamiltonian and the corresponding probability weight of random configurations, under which the method of gauge transformation is applicable and similar results to the Ising case can be obtained. Various random models are included in our framework. The system may have pair as well as many-body interactions and the symmetry group may be $O(n)$, Z_q, $SU(n)$ or any other groups satisfying conditions mentioned in the following sections. We show, in particular, that these general gauge glass systems do not have re-entrant transitions. This settles the issue of the existence of re-entrance in the XY gauge glass.

We introduce a general random system in section 2 and derive exact results. The internal energy and an upper bound on the specific heat are calculated exactly on a special line in the phase diagram; they are written in terms of a simple function even though the line crosses a phase boundary. Useful relations between correlation functions are also proved. These are all generalizations of results found in [10] for the $\pm J$ and Gaussian Ising spin glasses.

The topology of the phase diagram is discussed in section 3 based on the relations of correlation functions derived in section 2. In particular, we show that the present model does not have a re-entrant transition. One should note that our argument is not a mathematically rigorous proof of the absence of re-entrance because we use a plausible but unproved assumption of the existence of a ferromagnetic phase in a related (modified) model. This is a generalization of the Kitatani argument [11] for the absence of re-entrance in the $\pm J$ Ising model. The point is that the non-trivial problem of the absence of re-entrance in gauge glasses has been reduced to the trivial problem of the existence of a ferromagnetic phase in the modified model.

Experimentally, Ising-like spin glasses do not have re-entrance to a genuine spin glass phase without ferromagnetic (or antiferromagnetic) order [32]. The boundary between the spin glass and the antiferromagnetic states is vertical to the concentration axis in the phase diagram. This is in accordance with our conclusion as well as with the mean-field prediction [33]. Many isotropic Heisenberg spin glass materials also show no re-entrance to a simple spin glass phase [34–36] without ferromagnetic or antiferromagnetic order. Although our argument in the present paper does not apply immediately to such systems because the usual model of Heisenberg spin glasses does not have gauge symmetry, we may safely speculate that more elaborate models than a simple Edwards–Anderson model should be considered to explain the variety of complicated phase diagrams found experimentally (see [37]).

General arguments in sections 2 and 3 are applied to pair and many-body interactions in the remaining sections.

2. Gauge transformation in random spin systems

This section represents a generalization and reformulation of the theory developed in [10]. Our strategy is first to introduce a non-random system and then to go on to define a corresponding random model which incorporates gauge invariance. It may be useful to notice beforehand that the group element ϕ_i used below is a generalization of the Ising spin

variable S_i, and ω corresponds to J_{ij} of the $\pm J$ Ising model. It is important that the group of ω has a symmetry equal to or higher than that of ϕ_i (Z_2 for ϕ_i and ω in the $\pm J$ Ising model) for the whole argument to hold.

2.1. Phase space and non-random system

We restrict our attention to classical spin systems in the present paper. Let Λ be a set of N lattice points. We assign a spin variable $\phi_i \in \Phi$ with a non-negative measure $d\mu(\phi_i)$ at each site $i \in \Lambda$. The set Φ is the space of spin states at one lattice point. We denote $\{\phi\} \equiv (\phi_1, \ldots, \phi_N) \in \Phi^N$, a spin configuration of the total N-spin system. The measure of $\{\phi\}$ is expressed as $d\mu\{\phi\} = \prod_{i\in\Lambda} d\mu(\phi_i)$. For simplicity, we treat a system with a single coupling constant $J > 0$. Generalizations will be discussed later. The non-random Hamiltonian is denoted by

$$\mathcal{H}_0\{\phi\} = J\tilde{\mathcal{H}}_0\{\phi\} \tag{2.1}$$

where $\tilde{\mathcal{H}}_0\{\phi\}$ is the dimensionless part. The thermal average is expressed as

$$\langle\cdots\rangle_K \equiv Z_0(K)^{-1} \int d\mu\{\phi\} \cdots \exp\left(-K\tilde{\mathcal{H}}_0\{\phi\}\right) \tag{2.2}$$

where $K = \beta J > 0$, and

$$Z_0(K) = \int d\mu\{\phi\} \exp\left(-K\tilde{\mathcal{H}}_0\{\phi\}\right). \tag{2.3}$$

First, we assume the following condition on the set Φ and the measure $d\mu(\phi_i)$.

Condition I. There exists at least one kind of operation, denoted by $\phi \circ \psi$, such that Φ forms a group (Abelian or non-Abelian).

Then the group Φ is called a topological group [38–40]. The identity is denoted as ϕ_E and the inverse of ϕ is written as $\bar{\phi}$ in this group. For each $i \in \Lambda$, one can introduce a set of induced transformations, $U_\psi^{(i)}$ with $\psi \in \Phi$, in the space of functions of ϕ_i through the definition,

$$U_\psi^{(i)} : \phi_i \to \phi_i' = \phi_i \circ \bar{\psi}. \tag{2.4}$$

$U_\psi^{(i)}$ forms a group isomorphic to Φ for each $i \in \Lambda$ as

$$U_\theta^{(i)} U_\psi^{(i)} = U_{\theta\circ\psi}^{(i)} \tag{2.5}$$

for all θ, $\psi \in \Phi$. If Φ is non-Abelian, $U_\psi^{(i)}$ should be distinguished from

$$\tilde{U}_\psi^{(i)} : \phi_i \to \phi_i' = \psi \circ \phi_i \tag{2.6}$$

which is also isomorphic to Φ. The conjugation operator

$$U_c^{(i)} : \phi_i \to \phi_i' = \bar{\phi}_i \tag{2.7}$$

is also defined. The measure $\mathrm{d}\mu(\phi_i)$ is taken to be invariant under the transformations $U_\psi^{(i)}$, $\widetilde{U}_\psi^{(i)}$ and $U_{\mathrm{c}}^{(i)}$ for all $\psi \in \Phi$ in the sense that

$$\int \mathrm{d}\mu(\phi_i) U_\psi^{(i)} \cdots = \int \mathrm{d}\mu(\phi_i) \cdots \tag{2.8a}$$

$$\int \mathrm{d}\mu(\phi_i) \widetilde{U}_\psi^{(i)} \cdots = \int \mathrm{d}\mu(\phi_i) \cdots \tag{2.8b}$$

$$\int \mathrm{d}\mu(\phi_i) U_{\mathrm{c}}^{(i)} \cdots = \int \mathrm{d}\mu(\phi_i) \cdots \tag{2.8c}$$

or, equivalently,

$$\mathrm{d}\mu(\phi_i) = \mathrm{d}\mu(\phi_i \circ \psi) = \mathrm{d}\mu(\bar{\psi} \circ \phi_i) = \mathrm{d}\mu(\bar{\phi}_i). \tag{2.9}$$

If Φ is compact or Abelian, such an invariant measure exists [38–40]. We assume that Φ is compact and denote the volume by c

$$c \equiv \int \mathrm{d}\mu(\phi_i) < \infty. \tag{2.10}$$

The non-compact case is discussed in the appendix.

It is instructive to demonstrate the above equations in the case of the Ising system $\phi_i \in \{-1, 1\} = \Phi$. The operation $\phi \circ \psi$ is regarded as the product $\phi\psi$ with $\bar{\phi} = \phi$ and $\phi_{\mathrm{E}} = 1$. Then $U_\psi^{(i)}$ and $\widetilde{U}_\psi^{(i)}$ are equivalent, and $U_{\mathrm{c}}^{(i)}$ transforms $\phi_i \to -\phi_i$. Equations (2.8*a*) and (2.8*b*) mean $\sum_{\phi_i=\pm 1} f(\phi_i) = \sum_{\phi_i=\pm 1} f(\psi\phi_i) = f(1) + f(-1)$ for any function $f(\cdot)$ and $\psi = \pm 1$.

Three types of transformation in the space of functions of $\{\phi\}$ are introduced. These are the gauge transformation induced by $\{\psi\} \in \Phi^N$, the global transformation induced by $\psi \in \Phi$ and the global conjugation defined as

$$U_{\{\psi\}} \equiv \prod_{i \in \Lambda} U_{\psi_i}^{(i)} \tag{2.11a}$$

$$\widetilde{U}_\psi \equiv \prod_{i \in \Lambda} \widetilde{U}_\psi^{(i)} \tag{2.11b}$$

$$U_{\mathrm{c}} \equiv \prod_{i \in \Lambda} U_{\mathrm{c}}^{(i)} \tag{2.11c}$$

respectively. One finds that $U_{\{\psi\}}$ and $\widetilde{U}_\psi$ form groups isomorphic to Φ^N and Φ, respectively. From equation (2.8), the measure $\mathrm{d}\mu\{\phi\} = \Pi_i \mathrm{d}\mu(\phi_i)$ is invariant under the transformations $U_{\{\psi\}}$, $\widetilde{U}_\psi$ and U_{c} as

$$\int \mathrm{d}\mu\{\phi\} U_{\{\psi\}} \cdots = \int \mathrm{d}\mu\{\phi\} \cdots \tag{2.12a}$$

$$\int \mathrm{d}\mu\{\phi\} \widetilde{U}_\psi \cdots = \int \mathrm{d}\mu\{\phi\} \cdots \tag{2.12b}$$

$$\int \mathrm{d}\mu\{\phi\} U_{\mathrm{c}} \cdots = \int \mathrm{d}\mu\{\phi\} \cdots. \tag{2.12c}$$

In most interesting cases, the non-random Hamiltonian is invariant with respect to the global transformation $\widetilde{U}_\psi$ and the global conjugation U_{c}.

2.2. Random system

Let us consider N_R quenched random variables $\omega_n \in \Omega_n$ $(n = 1, 2, \ldots, N_R)$. The set Ω_n is the space of random configurations of ω_n, with a non-negative measure $d\nu_n(\omega_n)$. We denote $\{\omega\} \equiv (\omega_1, \ldots, \omega_{N_R}) \in \Omega_{tot}$, a random configuration of the total N_R-random variables, where $\Omega_{tot} \equiv \bigotimes_{n=1}^{N_R} \Omega_n$. The measure of $\{\omega\}$ is expressed as $d\nu\{\omega\} = \prod_{n=1}^{N_R} d\nu_n(\omega_n)$.

In order to construct a gauge-symmetric random system as a generalization of the non-random Hamiltonian (2.1), we introduce

$$\mathcal{H}\{\phi\}\{\omega\} = J\tilde{\mathcal{H}}\{\phi\}\{\omega\} \tag{2.13}$$

with the probability weight $P(K_p, \{\omega\})$, where the real parameter $K_p(\geqslant 0)$ controls the randomness. In equation (2.13), we explicitly indicate the dependence on the sets $\{\phi\}$ and $\{\omega\}$. The actual form of the Hamiltonian (2.13) will be specified for each problem so that the conditions below are satisfied. The thermal average $\langle\cdots\rangle_K$ and the partition function $Z(K, \{\omega\})$ are now expressed as in equations (2.2) and (2.3) with $\tilde{\mathcal{H}}_0\{\phi\}$ replaced by $\tilde{\mathcal{H}}\{\phi\}\{\omega\}$. The average over random configurations is defined by

$$[\cdots]_{K_p} \equiv \int d\nu\{\omega\} P(K_p, \{\omega\}) \cdots. \tag{2.14}$$

We introduce N_R kinds of local gauge transformations

$$V_{\{\psi\}}^{(n)} : \omega_n \to \omega_n' \in \Omega_n \tag{2.15}$$

in the space of functions of ω_n $(n = 1, 2, \ldots, N_R)$, each of which depends on the set $\{\psi\} \in \Phi^N$ and is homomorphic to Φ^N. The (global) gauge transformation defined by $V_{\{\psi\}} \equiv \prod_{n=1}^{N_R} V_{\{\psi\}}^{(n)}$ forms a group homomorphic to Φ^N. We assume that the following three conditions hold on $V_{\{\psi\}}$, $d\nu\{\omega\}$, $\mathcal{H}\{\phi\}\{\omega\}$ and $P(K_p, \{\omega\})$ for all $\{\psi\} \in \Phi^N$.

Condition II. The measure $d\nu\{\omega\}$ is an invariant measure in the sense that

$$\int d\nu\{\omega\} V_{\{\psi\}} \cdots = \int d\nu\{\omega\} \cdots. \tag{2.16}$$

A sufficient condition for equation (216) is that each $d\nu_n(\omega_n)$ is an invariant measure

$$\int d\nu_n(\omega_n) V_{\{\psi\}}^{(n)} \cdots = \int d\nu(\omega_n) \cdots. \tag{2.17}$$

Condition III. The Hamiltonian is invariant under any of $U_{\{\psi\}} V_{\{\psi\}}$ as

$$U_{\{\psi\}} V_{\{\psi\}} \mathcal{H}\{\phi\}\{\omega\} = \mathcal{H}\{\phi\}\{\omega\} \tag{2.18a}$$

or, equivalently,

$$U_{\{\psi\}} \mathcal{H}\{\phi\}\{\omega\} = V_{\{\bar{\psi}\}} \mathcal{H}\{\phi\}\{\omega\} \tag{2.18b}$$

where $\{\bar{\psi}\} = (\bar{\psi}_1, \bar{\psi}_2, \ldots, \bar{\psi}_N)$.

Condition IV. The probability weight $P(K_p, \{\omega\})$ is transformed by $V_{\{\psi\}}$ as

$$V_{\{\psi\}} P(K_p, \{\omega\}) = Y(K_p)^{-1} D\{\omega\} \exp\left(-K_p \tilde{\mathcal{H}}\{\psi\}\{\omega\}\right) \tag{2.19}$$

where $D\{\omega\}$ is an invariant function with respect to $V_{\{\psi\}}$.

Using equation (2.18*a*), equation (2.19) can be rewritten as

$$P(K_p, \{\omega\}) = Y(K_p)^{-1} D\{\omega\} \exp\left(-K_p \tilde{\mathcal{H}}\{\phi_E\}\{\omega\}\right) \tag{2.20}$$

where $\{\phi_E\} \in \Phi^N$ represents that $\phi_i = \phi_E$ for all $i \in \Lambda$. From the normalization condition $[1]_{K_p} = 1$, we find

$$Y(K_p) = \int d\nu\{\omega\} D\{\omega\} \exp\left(-K_p \tilde{\mathcal{H}}\{\phi_E\}\{\omega\}\right). \tag{2.21}$$

When $K_p = 0$, the probability weight becomes uniform (or independent of the random variables) except for the invariant part $D\{\omega\}$. It is natural to construct a model so that the non-random case is recovered in the limit of $K_p \to \infty$. See examples in later sections.

The following condition is required only in the discussion on the topology of the phase diagram. Anyway, most interesting cases satisfy this condition.

Condition V. The Hamiltonian is invariant under any global transformations $\tilde{U}_\psi$ as

$$\tilde{U}_\psi \mathcal{H}\{\phi\}\{\omega\} = \mathcal{H}\{\phi\}\{\omega\}. \tag{2.22}$$

As an example, the Hamiltonian of the $\pm J$ Ising model is a summation of local terms $-J\omega_{ij}\phi_i\phi_j$, where $\omega_{ij} \in \{-1, 1\} = \Phi$. The local gauge transformation $V_{\{\psi\}}^{(ij)}$ is defined by $\omega_{ij} \to \omega_{ij}\psi_i\psi_j$ with $\psi_i, \psi_j = \pm 1$. Equation (2.17) means $\sum_{\omega_{ij}=\pm 1} f(\omega_{ij}) = \sum_{\omega_{ij}=\pm 1} f(\omega_{ij}\psi_i\psi_j) = f(1) + f(-1)$ for any function $f(\cdot)$. Since each local term of the Hamiltonian, $-J\omega_{ij}\phi_i\phi_j$, is unchanged by the transformations $\phi_i \to \phi_i\psi_i$, $\phi_j \to \phi_j\psi_j$ and $\omega_{ij} \to \omega_{ij}\psi_i\psi_j$, it is easy to see the invariance of equation (2.18*a*). The probability weight of ω_{ij} is taken as $\exp(K_p\omega_{ij})/2\cosh K_p$. If p is defined by $\exp(-2K_p) = (1-p)/p$, this probability weight gives p if $\omega_{ij} = 1$ and $1-p$ otherwise, which is the usual weight of the $\pm J$ Ising model. The parameter K_p tends to infinity as $p \to 1$ (the non-random case) and to 0 as p approaches 1/2. The probability weight is transformed to $\exp(K_p\omega_{ij}\psi_i\psi_j)/2\cosh K_p$ by $V_{\{\psi\}}^{(ij)}$, and equation (2.19) holds with $D\{\omega\} = 1$.

2.3. Gauge invariance, internal energy and specific heat

We understand that each transformation operator acts on all functions following the operator. If the operator is rounded by brackets of any kind, it acts only within the brackets. From equations (2.16), (2.18) and (2.19), the configurational average of a function $Q\{\omega\}$ is expressed as

$$\begin{aligned}\left[Q\{\omega\}\right]_{K_p} &= \int d\nu\{\omega\} V_{\{\psi\}} P(K_p, \{\omega\}) Q\{\omega\} \\ &= Y(K_p)^{-1} \int d\nu\{\omega\} D\{\omega\} \exp\left(-K_p \tilde{\mathcal{H}}\{\psi\}\{\omega\}\right)\left(V_{\{\psi\}} Q\{\omega\}\right).\end{aligned} \tag{2.23}$$

Since the left-hand side of equation (2.23) is independent of $\{\psi\}$, we may integrate it by ψ and divide the result by the volume of the $\{\psi\}$-space:

$$\begin{aligned}\left[Q\{\omega\}\right]_{K_p} &= Y(K_p)^{-1}c^{-N}\int \mathrm{d}\mu\{\psi\}\int \mathrm{d}\nu\{\omega\}D\{\omega\}\exp\left(-K_p\tilde{\mathcal{H}}\{\psi\}\{\omega\}\right)\left(V_{\{\psi\}}Q\{\omega\}\right)\\ &= Y(K_p)^{-1}c^{-N}\int \mathrm{d}\nu\{\omega\}D\{\omega\}Z\left(K_p,\{\omega\}\right)\left\langle V_{\{\psi\}}Q\{\omega\}\right\rangle_{K_p}^{\{\psi\}} \qquad (2.24)\end{aligned}$$

where

$$\left\langle\cdots\right\rangle_{K_p}^{\{\psi\}} \equiv Z\left(K_p,\{\omega\}\right)^{-1}\int \mathrm{d}\mu\{\psi\}\exp\left(-K_p\tilde{\mathcal{H}}\{\psi\}\{\omega\}\right)\cdots. \qquad (2.25)$$

If $Q\{\omega\}$ is gauge invariant

$$V_{\{\psi\}}Q\{\omega\} = Q\{\omega\} \qquad (2.26)$$

for all $\psi \in \Phi^N$, equation (2.24) is rewritten as

$$\left[Q\{\omega\}\right]_{K_p} = Y(K_p)^{-1}c^{-N}\int \mathrm{d}\nu\{\omega\}D\{\omega\}Z\left(K_p,\{\omega\}\right)Q\{\omega\}. \qquad (2.27)$$

Using the invariance properties of equations (2.12*a*) and (2.18*a*), we find

$$\begin{aligned}V_{\{\psi\}}Z\left(K,\{\omega\}\right) &= \int \mathrm{d}\mu\{\phi\}U_{\{\psi\}}V_{\{\psi\}}\exp\left(-K\tilde{\mathcal{H}}\{\phi\}\{\omega\}\right)\\ &= \int \mathrm{d}\mu\{\phi\}\exp\left(-K\tilde{\mathcal{H}}\{\phi\}\{\omega\}\right) \qquad (2.28)\end{aligned}$$

which means that the partition function is gauge invariant. In general, the average of a function of the Hamiltonian, such as the internal energy E or the specific heat C, is gauge invariant from condition III.

The parameters K and K_p are independent of each other. However, if we impose the condition $K = K_p$, some exact results can be derived; the factor $Z\left(K_p,\{\omega\}\right)$ on the right-hand side of equation (2.27) is equal to the partition function $Z\left(K,\{\omega\}\right)$ appearing in the denominator of $Q\{\omega\}$ (if it is the thermal average of a physical quantity), and these two partition functions cancel out if $K = K_p$. The condition $K = K_p$ is equivalent to restricting ourselves to a subspace of the phase diagram, a generalization of the Nishimori line [10] of the $\pm J$ model, as will be shown later. Note that equation (2.21) is rewritten as

$$\begin{aligned}Y(K_p) &= \int \mathrm{d}\nu\{\omega\}V_{\{\psi\}}D\{\omega\}\exp\left(-K_p\tilde{\mathcal{H}}\{\phi_{\mathrm{E}}\}\{\omega\}\right)\\ &= c^{-N}\int \mathrm{d}\mu\{\psi\}\int \mathrm{d}\nu\{\omega\}D\{\omega\}\exp\left(-K_p\tilde{\mathcal{H}}\{\psi\}\{\omega\}\right)\\ &= c^{-N}\int \mathrm{d}\nu\{\omega\}D\{\omega\}Z\left(K_p,\{\omega\}\right). \qquad (2.29)\end{aligned}$$

Using equation (2.27), the internal energy is calculated as

$$\begin{aligned}E(K,K_p) &= \left[\left\langle\mathcal{H}\right\rangle_K\right]_{K_p} = -\left[Z\left(K,\{\omega\}\right)^{-1}\frac{\partial}{\partial\beta}Z\left(K,\{\omega\}\right)\right]_{K_p}\\ &= -JY(K_p)^{-1}c^{-N}\int \mathrm{d}\nu\{\omega\}D\{\omega\}\frac{Z\left(K_p,\{\omega\}\right)}{Z\left(K,\{\omega\}\right)}\frac{\partial}{\partial K}Z\left(K,\{\omega\}\right) \qquad (2.30a)\end{aligned}$$

which reduces to

$$E(K_p, K_p) = -JY(K_p)^{-1}c^{-N}\frac{\partial}{\partial K_p}\int d\nu\,\{\omega\}D\{\omega\}Z\big(K_p,\{\omega\}\big)$$
$$= -J\frac{\partial}{\partial K_p}\ln Y(K_p) \tag{2.30b}$$

if $K = K_p$. Similarly, the upper bound on the specific heat is estimated as

$$k_B T^2 C(K, K_p) = \Big[\langle\mathcal{H}^2\rangle_K\Big]_{K_p} - \Big[\langle\mathcal{H}\rangle_K^2\Big]_{K_p} \leqslant \Big[\langle\mathcal{H}^2\rangle_K\Big]_{K_p} - \Big[\langle\mathcal{H}\rangle_K\Big]_{K_p}^2$$
$$= \left[Z\big(K,\{\omega\}\big)^{-1}\frac{\partial^2}{\partial\beta^2}Z\big(K,\{\omega\}\big)\right]_{K_p} - \left[Z\big(K,\{\omega\}\big)^{-1}\frac{\partial}{\partial\beta}Z\big(K,\{\omega\}\big)\right]_{K_p}^2. \tag{2.31a}$$

If $K = K_p$, the same argument as above leads to

$$k_B T^2 C(K_p, K_p) \leqslant J^2 Y(K_p)^{-1}\frac{\partial^2}{\partial {K_p}^2}Y(K_p) - \left(JY(K_p)^{-1}\frac{\partial}{\partial K_p}Y(K_p)\right)^2$$
$$= J^2\frac{\partial^2}{\partial {K_p}^2}\ln Y(K_p). \tag{2.31b}$$

These results are easily modified when the origin and/or the scale of the energy are changed; when the Hamiltonian is changed as $\widetilde{\mathcal{H}}\{\phi\}\{\omega\} \to a + b\widetilde{\mathcal{H}}\{\phi\}\{\omega\}$ with arbitrary constants a and b, the above results hold by modifying them as $\ln Y(K_p) \to -aK_p + \ln Y(bK_p)$.

2.4. *Correlation functions*

Results in the present and next subsections will be useful for discussions on the topology of the phase diagram presented later. Let us consider an irreducible unitary representation of Φ (not the identity representation) denoted by $\gamma(\phi)$. The representation satisfies

$$\gamma(\phi_E) = E \tag{2.32a}$$

$$\gamma(\bar{\phi}) = \gamma^\dagger(\phi) \tag{2.32b}$$

where E is the identity matrix, and $\gamma^\dagger(\phi)$ denotes the adjoint matrix of $\gamma(\phi)$. Note that if Φ is Abelian, $\gamma(\phi)$ is a complex number. From the orthogonality of irreducible representations of a compact group [38–40], one finds

$$\int d\mu\,(\phi_i)\gamma(\phi_i) = 0. \tag{2.33}$$

Let us consider two types of two-point correlation functions,

$$f_\gamma(r; K, K_p) \equiv \Big[\mathrm{Tr}\langle\gamma(\bar{\phi}_0 \circ \phi_r)\rangle_K\Big]_{K_p} \tag{2.34a}$$

$$g_\gamma(r; K, K_p) \equiv \Big[\mathrm{Tr}\langle\gamma(\bar{\phi}_0 \circ \phi_r)\rangle_K\langle\gamma^\dagger(\bar{\phi}_0 \circ \phi_r)\rangle_K\Big]_{K_p} \tag{2.34b}$$

for $0, r \in \Lambda$. The operation $\mathrm{Tr}\cdots$ means the trace of the matrix, which plays no role if Φ is Abelian. In the case of the $\pm J$ Ising model ($\phi_i = \pm 1$), the correlation functions f_γ and g_γ correspond to the usual ferromagnetic correlation function $\left[\langle\phi_0\phi_r\rangle_K\right]_{K_p}$ and the spin glass correlation function $\left[\langle\phi_0\phi_r\rangle_K^2\right]_{K_p}$, respectively.

From equations (2.12*a*), (2.18) and (2.32), the thermal average $\langle\gamma(\bar\phi_0\circ\phi_r)\rangle_K$ is transformed as

$$\begin{aligned} V_{\{\psi\}}\langle\gamma(\bar\phi_0\circ\phi_r)\rangle_K &= Z\big(K,\{\omega\}\big)^{-1}\int \mathrm{d}\mu\,\{\phi\}V_{\{\psi\}}U_{\{\psi\}}\exp\big(-K\widetilde{\mathcal{H}}\{\phi\}\{\omega\}\big)\gamma(\bar\phi_0\circ\phi_r) \\ &= Z\big(K,\{\omega\}\big)^{-1}\int \mathrm{d}\mu\,\{\phi\}\exp\big(-K\widetilde{\mathcal{H}}\{\phi\}\{\omega\}\big)\big(U_{\{\psi\}}\gamma(\bar\phi_0\circ\phi_r)\big) \\ &= Z\big(K,\{\omega\}\big)^{-1}\int \mathrm{d}\mu\,\{\phi\}\exp\big(-K\widetilde{\mathcal{H}}\{\phi\}\{\omega\}\big)\gamma(\psi_0\circ\bar\phi_0\circ\phi_r\circ\bar\psi_r) \\ &= \gamma(\psi_0)\langle\gamma(\bar\phi_0\circ\phi_r)\rangle_K\gamma^\dagger(\psi_r). \end{aligned} \tag{2.35}$$

Using equations (2.24) and (2.35), we obtain

$$f_\gamma(r;K,K_p) = Y(K_p)^{-1}c^{-N}\int \mathrm{d}\nu\,\{\omega\}D\{\omega\}Z\big(K_p,\{\omega\}\big)\,\mathrm{Tr}\langle\gamma(\bar\phi_0\circ\phi_r)\rangle_K\langle\gamma^\dagger(\bar\phi_0\circ\phi_r)\rangle_{K_p}. \tag{2.36}$$

To simplify equation (2.36), we note that the product $\mathrm{Tr}\langle\gamma(\bar\phi_0\circ\phi_r)\rangle_K\langle\gamma^\dagger(\bar\phi_0\circ\phi_r)\rangle_{K'}$ is gauge invariant since from (2.35)

$$\begin{aligned} &V_{\{\psi\}}\,\mathrm{Tr}\langle\gamma(\bar\phi_0\circ\phi_r)\rangle_K\langle\gamma^\dagger(\bar\phi_0\circ\phi_r)\rangle_{K'} \\ &\quad= \mathrm{Tr}\,\gamma(\psi_0)\langle\gamma(\bar\phi_0\circ\phi_r)\rangle_K\gamma^\dagger(\psi_r)\gamma(\psi_r)\langle\gamma^\dagger(\bar\phi_0\circ\phi_r)\rangle_{K'}\gamma^\dagger(\psi_0). \end{aligned} \tag{2.37}$$

Therefore equations (2.27) and (2.36) yield

$$f_\gamma(r;K,K_p) = \Big[\mathrm{Tr}\langle\gamma(\bar\phi_0\circ\phi_r)\rangle_K\langle\gamma^\dagger(\bar\phi_0\circ\phi_r)\rangle_{K_p}\Big]_{K_p}. \tag{2.38}$$

Equation (2.38) leads to an important relation if $K = K_p$.

$$f_\gamma(r;K_p,K_p) = g_\gamma(r;K_p,K_p). \tag{2.39}$$

2.5. *Modified model*

Kitatani [11] showed by using a model with a slightly different probability weight from the usual $\pm J$ Ising model that this latter model does not have a re-entrant transition. We generalize his idea to the model introduced above. The Hamiltonian (2.13) remains the same in the modified model. The probability weight and the configurational average are replaced by

$$P_a\big(K_p,\{\omega\}\big) = P\big(K_p+a,\{\omega\}\big)\frac{Y(K_p+a)}{Y(K_p)}\frac{Z\big(K_p,\{\omega\}\big)}{Z\big(K_p+a,\{\omega\}\big)} \tag{2.40}$$

and

$$\{\cdots\}^a_{K_p} \equiv \int d\nu\,\{\omega\} P_a(K_p,\{\omega\})\cdots \tag{2.41}$$

with a fixed real value a. The normalization condition $\{1\}^a_{K_p} = 1$ is satisfied; from equations (2.27), (2.29) and (2.40),

$$\int d\nu\,\{\omega\} P_a(K_p,\{\omega\}) = \frac{Y(K_p+a)}{Y(K_p)}\left[\frac{Z(K_p,\{\omega\})}{Z(K_p+a,\{\omega\})}\right]_{K_p+a}$$

$$= Y(K_p)^{-1}c^{-N}\int d\nu\,\{\omega\} D\{\omega\} Z(K_p,\{\omega\}) = 1. \tag{2.42}$$

We call this model the modified model with a.

According to equations (2.19) and (2.40), the probability weight $P_a(K_p,\{\omega\})$ is transformed as

$$V_{\{\psi\}} P_a(K_p,\{\omega\}) = Y(K_p)^{-1} D\{\omega\}\frac{Z(K_p,\{\omega\})}{Z(K_p+a,\{\omega\})}\exp\left(-(K_p+a)\tilde{\mathcal{H}}\{\psi\}\{\omega\}\right). \tag{2.43}$$

Then, similarly to equation (2.24), the configurational average of a function $Q\{\omega\}$ is expressed by

$$\{Q\{\omega\}\}^a_{K_p} = \int d\nu\,\{\omega\} V_{\{\psi\}} P_a(K_p,\{\omega\}) Q\{\omega\}$$

$$= Y(K_p)^{-1}\int d\nu\,\{\omega\} D\{\omega\}\frac{Z(K_p,\{\omega\})}{Z(K_p+a,\{\omega\})}$$

$$\times \exp\left(-(K_p+a)\tilde{\mathcal{H}}\{\psi\}\{\omega\}\right)\left(V_{\{\psi\}} Q\{\omega\}\right)$$

$$= Y(K_p)^{-1}c^{-N}\int d\nu\,\{\omega\} D\{\omega\} Z(K_p,\{\omega\})\left(V_{\{\psi\}} Q\{\omega\}\right)^{\{\psi\}}_{K_p+a} \tag{2.44}$$

where $\{\cdots\}^{\{\psi\}}_{K_p}$ is defined by equation (2.25). If $Q\{\omega\}$ is gauge invariant, $V_{\{\psi\}} Q\{\omega\}$ $(= Q\{\omega\})$ is independent of $\{\psi\}$. In this case we find from equations (2.27) and (2.44)

$$\{Q\{\omega\}\}^a_{K_p} = [Q\{\omega\}]_{K_p}. \tag{2.45}$$

Equation (2.45) shows that all gauge invariant quantities are independent of a, and equal to those in the original model (which corresponds to the $a = 0$ case).

Let us consider the correlation functions

$$f^a_\gamma(r; K, K_p) \equiv \left\{\mathrm{Tr}\langle\gamma(\bar{\phi}_0\circ\phi_r)\rangle_K\right\}^a_{K_p} \tag{2.46a}$$

$$g^a_\gamma(r; K, K_p) \equiv \left\{\mathrm{Tr}\langle\gamma(\bar{\phi}_0\circ\phi_r)\rangle_K\langle\gamma^\dagger(\bar{\phi}_0\circ\phi_r)\rangle_K\right\}^a_{K_p}. \tag{2.46b}$$

Using equations (2.35) and (2.44), we obtain

$$f^a_\gamma(r; K, K_p) = Y(K_p)^{-1}c^{-N}\int d\nu\,\{\omega\} D\{\omega\} Z(K_p,\{\omega\})\,\mathrm{Tr}\langle\gamma(\bar{\phi}_0\circ\phi_r)\rangle_K\langle\gamma^\dagger(\bar{\phi}_0\circ\phi_r)\rangle_{K_p+a} \tag{2.47}$$

which is rewritten from equation (2.27) as

$$f_\gamma^a(r; K, K_p) = \left[\mathrm{Tr}\langle \gamma(\bar{\phi}_0 \circ \phi_r)\rangle_K \langle \gamma^\dagger(\bar{\phi}_0 \circ \phi_r)\rangle_{K_p+a} \right]_{K_p}. \tag{2.48}$$

The product $\mathrm{Tr}\langle \gamma(\bar{\phi}_0 \circ \phi_r)\rangle_K \langle \gamma^\dagger(\bar{\phi}_0 \circ \phi_r)\rangle_{K'}$ is gauge invariant as is verified from the transformation property in equation (2.37). Since gauge invariant quantities have the same value for any a as shown in equation (2.45), we find

$$f_\gamma^a(r; K, K_p) = \left\{ \mathrm{Tr}\langle \gamma(\bar{\phi}_0 \circ \phi_r)\rangle_K \langle \gamma^\dagger(\bar{\phi}_0 \circ \phi_r)\rangle_{K_p+a} \right\}^a_{K_p}. \tag{2.49}$$

Comparison of equations (2.46*b*) and (2.49) yields

$$f_\gamma^a(r; K_p + a, K_p) = g_\gamma^a(r; K_p + a, K_p) \tag{2.50}$$

under the condition $K = K_p + a$.

Relations of correlation functions between the modified model and the original model are derived from equations (2.45), (2.48) and (2.38) as

$$g_\gamma^a(r; K, K_p) = g_\gamma(r; K, K_p) \tag{2.51}$$

and

$$f_\gamma^a(r; K_p, K_p) = f_\gamma(r; K_p + a, K_p)^* \tag{2.52}$$

where the asterisk denotes the complex conjugate. Applying equation (2.48) to two distinct modified models with a and b with equation (2.45) taken into account, we find that the following relations hold between two modified models with a and b,

$$g_\gamma^a(r; K, K_p) = g_\gamma^b(r; K, K_p) \tag{2.53}$$

and

$$f_\gamma^a(r; K_p + b, K_p) = f_\gamma^b(r; K_p + a, K_p)^*. \tag{2.54}$$

2.6. Multi-coupling model

The present theory can be generalized to accommodate multi-spin couplings,

$$\mathcal{H}_0\{\phi\} = -\sum_n J^{(n)} \tilde{\mathcal{H}}_0^{(n)}. \tag{2.55}$$

The dimensionless constants will be written in a vector notation $\boldsymbol{K} = (K^{(1)}, K^{(2)}, \cdots)$, where $K^{(n)} = \beta J^{(n)}$. The probability weight has a set of corresponding parameters $\boldsymbol{K}_p = (K_p^{(1)}, K_p^{(2)}, \ldots)$. By similar arguments as above, the internal energy and the specific heat are obtained under the condition $\boldsymbol{K} = \boldsymbol{K}_p$ as

$$E(\boldsymbol{K}_p, \boldsymbol{K}_p) = -\sum_n J^{(n)} \frac{\partial}{\partial K_p^{(n)}} \ln Y(\boldsymbol{K}_p) \tag{2.56a}$$

$$k_B T^2 C(\boldsymbol{K}_p, \boldsymbol{K}_p) \leqslant \sum_n (J^{(n)})^2 \left(\frac{\partial}{\partial K_p^{(n)}} \right)^2 \ln Y(\boldsymbol{K}_p). \tag{2.56b}$$

All results in previous subsections hold by replacing K and K_p with $\boldsymbol{K}$ and $\boldsymbol{K}_p$.

3. Long-range order and topology of the phase diagram

3.1. Order parameters

We assume that condition V in section 2 holds, which requires a global symmetry of the Hamiltonian, in order to define the ordered phases. From condition (2.33) with equation (2.22), we find

$$\begin{aligned}\langle\gamma(\phi_i)\rangle_K &= Z(K,\{\omega\})^{-1}\int \mathrm{d}\mu\{\phi\}\tilde{U}_\psi \exp\bigl(-K\tilde{\mathcal{H}}\{\phi\}\{\omega\}\bigr)\gamma(\phi_i)\\ &= Z(K,\{\omega\})^{-1}\int \mathrm{d}\mu\{\phi\}\exp\bigl(-K\tilde{\mathcal{H}}\{\phi\}\{\omega\}\bigr)\gamma(\psi\circ\phi_i)\\ &= Z(K,\{\omega\})^{-1}\int \mathrm{d}\mu\{\phi\}\exp\bigl(-K\tilde{\mathcal{H}}\{\phi\}\{\omega\}\bigr)c^{-1}\int \mathrm{d}\mu(\psi)\gamma(\psi\circ\phi_i)\\ &= 0 \end{aligned} \tag{3.1}$$

for all $i \in \Lambda$. Equation (3.1) means that the Gibbs measure of a configuration $\{\omega\}$, $\mathrm{d}\mu\{\phi\}\exp\bigl(-K\tilde{\mathcal{H}}\{\phi\}\{\omega\}\bigr)/Z(K,\{\omega\})$, has the global symmetry and consequently there is no spontaneous symmetry breaking in finite systems. A comment on a symmetry-breaking field is in order. If an external field such as

$$\beta\mathcal{H}_L\{\phi\} = -L\sum_{i\in\Lambda}\mathrm{Tr}\bigl(\gamma(\phi_i)+\gamma^\dagger(\phi_i)\bigr) \tag{3.2}$$

is added to the Hamiltonian, the global symmetry of equation (2.22) and the symmetry $\langle\gamma(\phi_i)\rangle_K = 0$ are simultaneously broken. Note that the system with the external field (3.2) is not gauge-symmetric since it does not satisfy conditions III and IV. Thus we treat the case of $L = 0$ in this paper.

It is understood throughout this section that the thermodynamic limit ($N \to \infty$) has already been taken in the correlation functions f_γ and g_γ. The order parameters are defined by

$$m_\gamma(K,K_p) \equiv \overline{\lim_{r\to\infty}}\,\bigl|f_\gamma(r;K,K_p)\bigr| \tag{3.3a}$$

$$q_\gamma(K,K_p) \equiv \overline{\lim_{r\to\infty}}\, g_\gamma(r;K,K_p). \tag{3.3b}$$

Finiteness of m_γ indicates the existence of ferromagnetic long-range order. In the $\pm J$ Ising model, m_γ is equal to the long-range limit ($r \to \infty$) of the correlation function $[\langle\phi_0\phi_r\rangle_K]_{K_p}$. The second quantity q_γ corresponds to the spin glass order parameter, $[\langle\phi_0\phi_r\rangle_K^2]_{K_p}$ in the limit $r \to \infty$, of the $\pm J$ Ising model. In both of these order parameters, it is not necessarily mathematically trivial that finiteness of long-range order (3.3) readily implies symmetry breaking defined by finiteness of a one-point function in the infinitesimal-field limit. Although equivalence of these two concepts is usually taken for granted, our discussions in this paper will be made only in terms of long-range order (3.3) to avoid confusion.

The same order parameters as in equation (3.3) are defined in the modified model. For the ferromagnetic order parameter,

$$m_\gamma^a(K,K_p) \equiv \overline{\lim_{r\to\infty}}\,\bigl|f_\gamma^a(r;K,K_p)\bigr|. \tag{3.4}$$

The order parameter q_γ is identical to the original case ($a = 0$), since it is gauge invariant, see equation (2.45). Two kinds of ordered phase can be considered;

$$m_\gamma = 0 \qquad q_\gamma = 0 \qquad \text{(paramagnetic phase)} \tag{3.5a}$$

$$m_\gamma > 0 \qquad q_\gamma > 0 \qquad \text{(ferromagnetic phase)} \tag{3.5b}$$

$$m_\gamma = 0 \qquad q_\gamma > 0 \qquad \text{(glass phase).} \tag{3.5c}$$

The case $m_\gamma > 0$ and $q_\gamma = 0$ is unlikely to exist and will be excluded hereafter.

If the non-random system has no long-range order but has Kosterlitz–Thouless (KT)-like topological order [41, 42], one may define correlation lengths as follows

$$\xi(K, K_p) = \overline{\lim_{r\to\infty}} \left| \frac{\partial}{\partial r} \ln |f_\gamma(r; K, K_p)| \right|^{-1} \tag{3.6a}$$

$$\tilde{\xi}(K, K_p) = \overline{\lim_{r\to\infty}} \left| \frac{\partial}{\partial r} \ln g_\gamma(r; K, K_p) \right|^{-1} \tag{3.6b}$$

with

$$\xi < +\infty \qquad \tilde{\xi} < +\infty \qquad \text{(paramagnetic phase)} \tag{3.7a}$$

$$\xi = +\infty \qquad \tilde{\xi} = +\infty \qquad \text{(uniform KT phase)} \tag{3.7b}$$

$$\xi < +\infty \qquad \tilde{\xi} = +\infty \qquad \text{(random KT phase).} \tag{3.7c}$$

In the uniform KT phase there is no long-range order and the correlation functions $f_\gamma(r)$ and $g_\gamma(r)$ decay in power laws. The random KT phase has an exponentially decaying ferromagnetic correlation $f_\gamma(r)$ and a power-decaying $g_\gamma(r)$.

Note that we are discussing the possibilities of various ordered phases, not proving their existence. The arguments in the following sections are about the possible shapes of the phase boundary of the ferromagnetic (or uniform KT) phase when this phase occupies a finite region in the phase diagram. The existence of a glass (or a random KT) phase will not be necessary.

3.2. Topology of the phase diagram

We discuss the topology of the phase diagram in the K_p–K plane around the multicritical point. We treat the case with the following properties.

(i) There exist at least two phases, paramagnetic and ferromagnetic.

(ii) The system is paramagnetic when K is sufficiently small.

(iii) The system is ferromagnetic when K and K_p are sufficiently large, including the region with large but finite K in the non-random case ($K_p = \infty$) and large but finite K_p in the ground state ($K = \infty$).

We draw the phase diagram with the axes of K_p and K. The boundary between the ordered and paramagnetic phases (OP boundary) marks the onset of finite q_γ since both ordered phases in equation (3.5) have $q_\gamma > 0$ (figure 1). We express this boundary line by its intersection with the line $K = K_p + x$ as

$$(K, K_p) = (K_c(x) + x, K_c(x)). \tag{3.8}$$

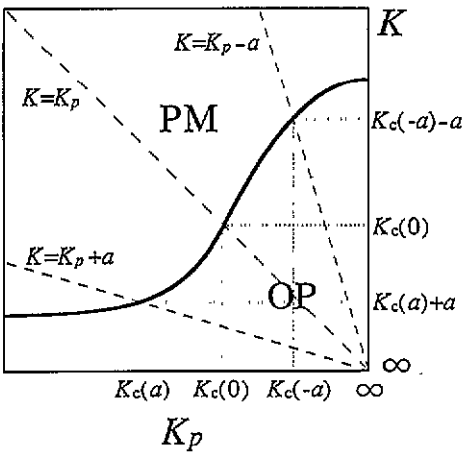

Figure 1. Schematic phase diagram in the K_p–K plane. The right bottom point has $(K_p, K) = (\infty, \infty)$, the zero-temperature limit of the non-random system. The value of K_p increases from left to right along the K_p axis and that of K increases from top to bottom along the K axis. The bold curve is the OP boundary, the boundary between the ordered phase (OP) and the paramagnetic phase (PM). The three broken lines indicating $K = K_p$, $K = K_p + a$ and $K = K_p - a$ are typical examples for the definition of the location of the OP boundary, equation (3.8).

The position of a point on the OP boundary (the bold curve in figure 1) is parametrized by x. For instance, $x = 0$ means the point with $K = K_p$ since $K = K_c(0)$ and $K_p = K_c(0)$ from equation (3.8). If $x > 0$, $K = K_p + x$ means $K > K_p$ and the corresponding point on the OP boundary lies below the line $K = K_p$ (the point with the K axis $K = K_c(a) + a$ in figure 1). A point above the line $K = K_p$ on the phase diagram has a negative x. Our discussions will be restricted to the region around $K_p = K_c(0)$ which is a multicritical region as will be seen later.

First, we consider the original model ($a = 0$ in the modified model). Taking the limit of $r \to \infty$ of the absolute values of both sides of equation (2.39), we obtain

$$m_\gamma(K_p, K_p) = q_\gamma(K_p, K_p) \tag{3.9}$$

from the definition (3.3) of the order parameters. This implies the absence of the glass phase on the line $K = K_p$ in the phase diagram since the glass phase has $q_\gamma > m_\gamma = 0$. In other words, the line $K = K_p$ does not enter the glass phase (if any). Thus, on $K = K_p$, the system is ferromagnetic on the low-temperature side $K_p > K_c(0)$, and paramagnetic if $K_p < K_c(0)$.

Next, we show that no ferromagnetic phase exists for any K (any temperature) in the region $K_p < K_c(0)$ in the phase diagram of the original model (the region on the left-hand side of the vertical dotted line at $K_p = K_c(0)$ in figures 1 and 2) by comparing its behaviour with that of the modified model. Taking the limit of $r \to \infty$ on both sides of equation (2.52), we obtain

$$m_\gamma^a(K_p, K_p) = m_\gamma(K_p + a, K_p). \tag{3.10}$$

Let us first remark that the shape of the OP boundary is independent of the parameter a since the OP boundary is defined in terms of q_γ which does not depend upon a from gauge invariance. Thus the OP boundary of the modified model is parametrized as in equation (3.8) for any a. By definition, the modified model with a is paramagnetic above the OP boundary. Thus $m_\gamma^a(K = K_p, K_p) = 0$ when $K_p < K_c(0)$ since $K = K_p < K_c(0)$ implies considering the left-hand portion of the line $K = K_p$ lying above the OP boundary (marked with dashes and dots in figure 2(*b*)). According to equation (3.10), $m_\gamma^a(K_p, K_p) = 0$ leads to $m_\gamma(K_p + a, K_p) = 0$. This latter relation indicates that the original model is not ferromagnetic when $K = K_p + a$ with $K_p < K_c(0)$. The condition $K = K_p + a$ with

$K_p < K_c(0)$ corresponds to the left-hand portion of the line $K = K_p + a$ denoted in a chain curve in figure 2(a). Since the above argument holds for any a, we may conclude that the original model is not ferromagnetic at any point with $K_p < K_c(0)$ (region on the left-hand side of the dotted line at $K_p = K_c(0)$ in figure 2(a)). Consequently, the boundary between the ferromagnetic and glass (or paramagnetic) phases (to be called the F boundary hereafter) does not lie in the region $K_p < K_c(0)$. Allowed shapes of the F boundary are schematically indicated in figure 3(a) by (v) (vertical) and (r) (re-entrant).

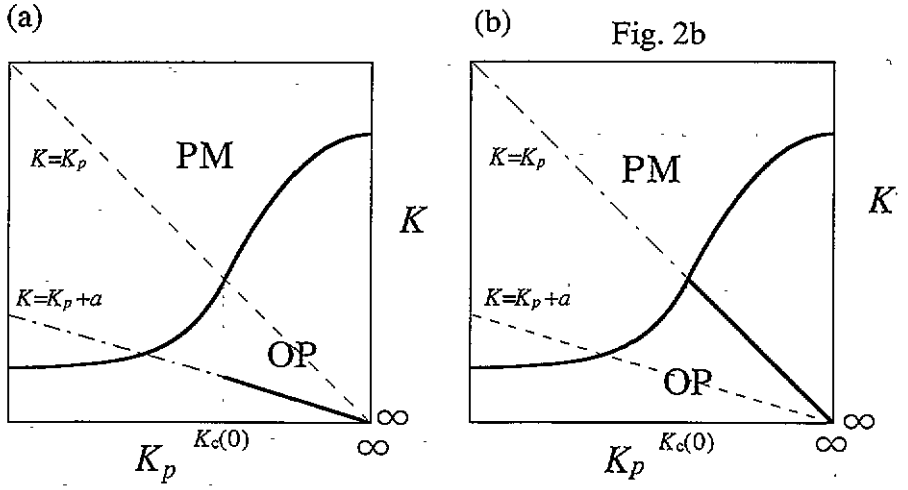

Figure 2. The K_p–K plane introduced in figure 1 for (a) the original model and (b) the modified model with $a > 0$. The ferromagnetic order parameter on the line $K = K_p + a$ in the original model is identical to that on $K = K_p$ in the modified model; both are indicated by chain and bold lines in (a) and (b). The chain part indicates the region where the system should not be ferromagnetic. The bold part indicates the region where the system is allowed to be ferromagnetic.

The same argument shows that the F boundary of the modified model with any a is either vertical or re-entrant. A generalization of equation (3.10) is

$$m_\gamma^b(K_p + a, K_p) = m_\gamma^a(K_p + b, K_p). \tag{3.11}$$

which is derived from equation (2.54). We note that the line $K = K_p + a$ does not enter the glass phase ($q_\gamma^a > m_\gamma^a = 0$) because $m_\gamma^a = q_\gamma^a$ when $K = K_p + a$ from equation (2.50). Hence the ordered phase on the line $K = K_p + a$ should be ferromagnetic. By replacing x in equation (3.8) with a, we recognize that the modified model with b on the line $K = K_p + a$ (whose order parameter is on the left-hand side of equation (3.11)) is critical at $K_p = K_c(a)$. Consequently $m_\gamma^b(K_p + a, K_p)$ is zero for $K_p < K_c(a)$. Equation (3.11) immediately yields that the modified model with a is not ferromagnetic on the line $K = K_p + b$ with $K_p < K_c(a)$. This last result holds for any b. Therefore the F boundary of the modified model does not lie on the left-hand region of the vertical line at $K_p = K_c(a)$. The allowed F boundaries are indicated in figure 3(b) as (v) (vertical) and (r) (re-entrant).

For the $\pm J$ Ising model, Kitatani [11] developed an argument to show that the F boundary is, in fact, vertical (indicated by (v) in figure 3(a)) to the K_p axis in the K_p–K plane in the original model, which means the absence of re-entrant transitions (indicated

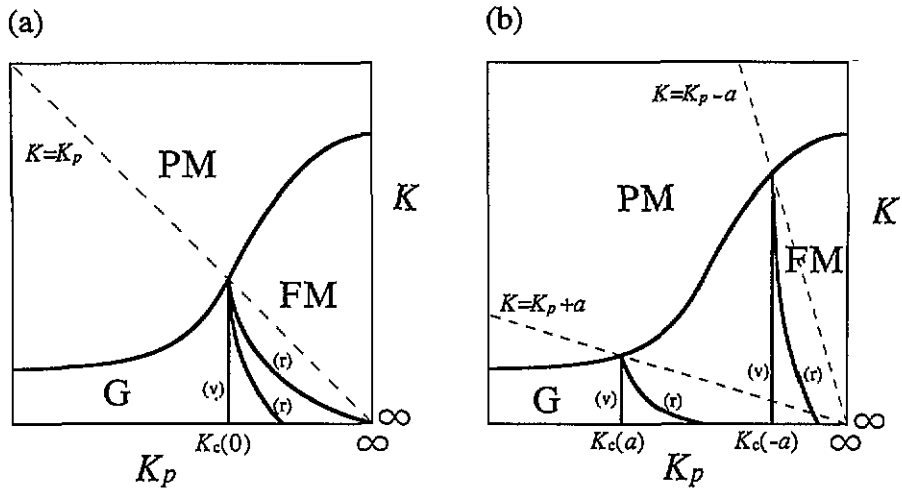

Figure 3. The K_p–K phase diagram in the case that paramagnetic (PM), ferromagnetic (FM) and glass (G) phases appear for (*a*) the original model, (*b*) the modified model with $a > 0$ and $-a < 0$. Possible F boundaries are indicated as vertical (v) and re-entrant (r).

by (r) in figure 3(*a*)). We apply his method to the present general model. It is natural to assume that the ordered phase of the region $K < K_p + a$ (above the line $K = K_p + a$) is ferromagnetic, not a glass, in the modified model with a, because this regime is located between the lines $K = K_p + a$ and $K_p = \infty$ (the non-random limit), both of which are ferromagnetic for large $K (> K_c(x) + x)$. This is a reasonable but unproved assumption. We thus refrain from claiming that the present argument is a final proof of the absence of re-entrance.

Since the line $K = K_p$ lies in this region above $K = K_p + a$ if $a > 0$ (figure 2(*b*)), $m_\gamma^a(K_p, K_p) > 0$ holds when $K_p > K_c(0)$ on the line $K = K_p$ (indicated by the bold curve in figure 2(*b*)). This implies, from equation (3.10), $m_\gamma(K_p + a, K_p) > 0$ if and only if $K_p > K_c(0)$ (indicated by the bold curve in figure 2(*a*)). In other words, the ferromagnetic phase extends from $K_p = \infty$ to $K_p = K_c(0)$ on the line $K = K_p + a$ in the original model. Since this argument holds for any a, we conclude that the ferromagnetic order parameter is finite below the OP boundary at any K_p satisfying $K_p > K_c(0)$. The F boundary is thus vertical. This fact was anticipated from various analyses [5–7, 12].

We have derived various properties of the gauge-symmetric model which satisfies conditions I–V. If the system has the three phases defined in equation (3.5), the phase diagram has the generic topology shown in figure 3(*a*); the F boundary is vertical at $K_p = K_c(0)$ and intersects the OP boundary and the line $K = K_p$ at the same point, a multicritical point. Note that the present theory gives no information on the existence of the glass phase or the precise locations of the critical points. One should employ other methods, such as Monte Carlo simulations, to answer these questions. If the glass phase does not exist, the vertical F boundary separates the ferromagnetic and paramagnetic phases. This holds because we have not used the existence of a glass phase in the above argument.

It is possible to apply the above argument to the modified model with a using equation (3.11) with $b > a$. It is natural to consider that the ordered phase of the region $K < K_p + b$ above the line $K = K_p + b$ is ferromagnetic, not a glass, in the modified model with $b (> a)$, because this regime is located between the lines $K = K_p + b$ and $K_p = \infty$, both of which are ferromagnetic at large K. Since the line $K = K_p + a$ lies in this region if

$b > a$, $m_\gamma^b(K_p + a, K_p) > 0$ holds when $K_p > K_c(a)$ (the right-hand region of the vertical line at $K_p = K_c(a)$). This implies, from equation (3.11), $m_\gamma^a(K_p + b, K_p) > 0$ if and only if $K_p > K_c(a)$. We find that the F boundary in any modified model is vertical to the K_p axis in the K_p–K plane; the line (v) in figure 3 is realized.

The modified model has the following interesting property. As has been pointed out by Kitatani [11] in the special case of the $\pm J$ Ising model, it is possible to construct a model with a glass phase in any dimension if there exists a finite region of the ferromagnetic phase. More precisely, a modified model with *negative* a has $K_c(a) > K_c(0)$ (figure 1). This means that the ordered phase in the range $K_c(0) < K_p < K_c(a)$ is not ferromagnetic since $K_c(a)$ marks the lower bound of the ferromagnetic phase of the modified model with a.

The above argument gives an explicit example in which the lower critical dimension of the glass phase is equal to that of the existence of a finite region of the ferromagnetic phase. The only restriction is that the distribution of randomness should follow that of the modified model (2.40) with negative a.

3.3. Kosterlitz–Thouless phase and other possibilities

If the non-random system has no long-range order but has KT-like topological order, one may define the ordered phases using the correlation lengths in equation (3.6). Then a similar argument as above can be made for the three phases in equation (3.6). It suffices to replace finiteness of m_γ and m_γ^a in the preceding paragraph by divergence of ξ in equation (3.6*a*) and the corresponding ξ^a from the modified model. It should also be remembered that equation (2.50) guarantees that the divergence of the usual ferromagnetic correlation length ξ^a is equivalent to that of the glass correlation length $\tilde{\xi}^a$ if $K = K_p + a$ in the modified model. This means that the line $K = K_p + a$ does not enter the random KT phase (if any). The function $K_c(x)$ is again defined as the boundary of the paramagnetic phase. The conclusion is that the boundary of the uniform KT phase (KT boundary) is shown to be vertical to the K_p axis below a multicritical point; it is at $K_p = K_c(0)$ in the original model and at $K_p = K_c(a) + a$ in the modified model with a (figure 4).

If the non-random system has both KT and ferromagnetic phases, then the paramagnetic, KT and ferromagnetic phases appear in sequence as K increases. It is necessary to modify equation (3.5*a*) as

$$m_\gamma = 0 \qquad q_\gamma = 0 \qquad \text{(paramagnetic or KT phase)}. \tag{3.12}$$

The arguments presented so far apply to both the KT and F boundaries. We conclude the verticality of both boundaries as shown in figure 5.

The fact that the non-glass ordered phase is ferromagnetic (or KT-type) is not essential in the above argument for the topology of the phase diagram. If the non-random system has a different type of ordered phase, whose order parameter can be defined from $f_\gamma(r; K, K_p)$, the same argument can be made for the paramagnetic, gauge glass and new phases. If there are several types of long-range order, it is necessary to introduce other irreducible representations γ_1, γ_2 etc. Then a similar argument can be developed as above.

4. Applications to pair interactions I

In the present section, we consider a system of pair interactions with local random variables associated with each pair. Let $\Lambda_B \subset \Lambda \otimes \Lambda$ be a set of N_B pairs of sites in Λ. The

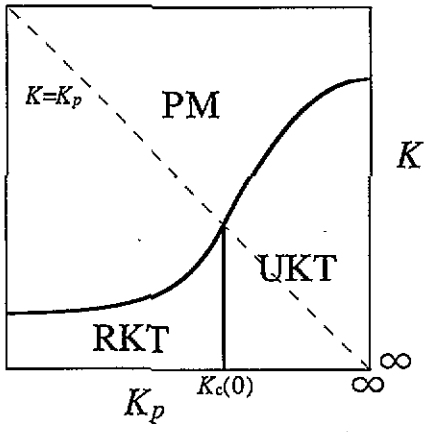

Figure 4. The K_p–K phase diagram is shown in the case that the system has paramagnetic (PM), uniform KT (UKT) and random KT (RKT) phases. The bold curve is the OP boundary. The vertical bold line at $K_p = K_c(0)$ is the KT boundary.

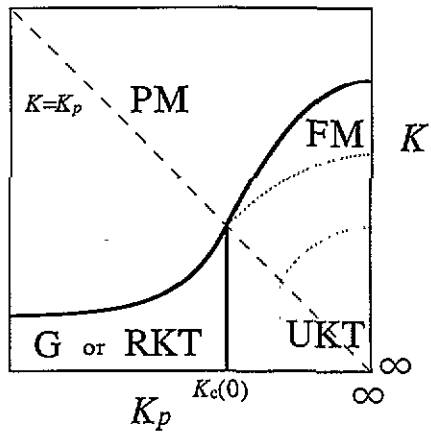

Figure 5. The K_p–K phase diagram is shown for the case in which the system has paramagnetic, uniform KT, random KT, ferromagnetic and glass phases. The bold curve is the OP boundary. The vertical bold line at $K_p = K_c(0)$ is the F boundary. Two possibilities for the boundary of the uniform KT phase are indicated by the dotted curves; in one of these, the KT boundary is identical to the F boundary at $K_p = K_c(0)$.

non-random Hamiltonian is assumed to have the form

$$\mathcal{H}_0\{\phi\} = -J \sum_{\langle ij\rangle \in \Lambda_B} \varepsilon(\phi_i \circ \bar{\phi}_j) \tag{4.1}$$

where a pair $\langle ij\rangle$ is taken only once in the summation. Since the function $\varepsilon(\phi)$ should be real, it is necessary that

$$\varepsilon(\phi) = \varepsilon(\bar{\phi}) \tag{4.2}$$

for all $\phi \in \Phi$. Further, we assume

$$\varepsilon(\phi \circ \psi) = \varepsilon(\psi \circ \phi) \tag{4.3}$$

for all $\phi, \psi \in \Phi$. Equation (4.3), which is trivial if Φ is Abelian, is needed for the global symmetry of the random Hamiltonian, equation (2.22), to be introduced later. Equation (4.3) is satisfied if $\varepsilon(\phi)$ is a function of characters of representations of Φ.

We introduce a class of gauge-symmetric models associated with the non-random Hamiltonian (4.1)–(4.3). We assign a random variable $\omega_{ij} \in \Phi$ to each bond $\langle ij\rangle$. The space Φ to which ω_{ij} belongs and the measure $d\mu(\omega_{ij})$ are chosen to be the same as those for the spin variables. Thus $\Omega_{tot} = \Phi^{N_B}$, and $d\nu\{\omega\} = \prod_{\langle ij\rangle \in \Lambda_B} d\mu(\omega_{ij})$. The Hamiltonian and the probability weight are chosen to be

$$\mathcal{H}\{\phi\}\{\omega\} = -J \sum_{\langle ij\rangle \in \Lambda_B} \varepsilon(\phi_i \circ \omega_{ij} \circ \bar{\phi}_j) \tag{4.4}$$

and

$$P\big(K_p, \{\omega\}\big) = \prod_{\langle ij\rangle \in \Lambda_B} I(K_p)^{-1} \exp\big(K_p \varepsilon(\omega_{ij})\big) \tag{4.5}$$

where $I(K_p)$ is a function determined by the normalization. The gauge transformation $V_{\{\psi\}} = \prod_{(ij)\in\Lambda_B} V^{(ij)}_{\psi_i,\psi_j}$ is constructed as the local transformation

$$V^{(ij)}_{\psi_i,\psi_j} : \omega_{ij} \to \omega'_{ij} = \psi_i \circ \omega_{ij} \circ \bar{\psi}_j \tag{4.6}$$

by assuming $i > j$. Ordering site numbers is necessary since, in general, the transformation (4.6) is different from $\psi_j \circ \omega_{ij} \circ \bar{\psi}_i$. Note that $V^{(ij)}_{\psi_i,\psi_j}$ forms a group homomorphic to Φ^2, and $V_{\{\psi\}}$ is homomorphic to Φ^N. Thus it is easy to see that the above system satisfies conditions I to V. Consequently, if the non-random Hamiltonian is expressed by equation (4.1) with equations (4.2) and (4.3), one can derive an associated gauge-symmetric random model of equations (4.4) and (4.5) for any Φ, Abelian or non-Abelian. Note that equations (4.4) and (4.5) are not the only possible way to derive a gauge-symmetric random model from equations (4.1)–(4.3). One of the other possibilities is discussed in the next section.

From equations (2.21) and (4.5), we obtain the normalization factor as

$$Y(K_p) = I(K_p)^{N_B} \tag{4.7}$$

$$I(K_p) = \int \mathrm{d}\mu(\omega_{ij}) \exp\left(K_p \varepsilon(\omega_{ij})\right). \tag{4.8}$$

The general result of equations (2.30*b*) and (2.31*b*) can be written with equation (4.7) as

$$E(K_p, K_p) = -N_B J \frac{\partial}{\partial K_p} \ln I(K_p) \tag{4.9}$$

$$k_B T^2 C(K_p, K_p) \leqslant N_B J^2 \frac{\partial^2}{\partial {K_p}^2} \ln I(K_p) \tag{4.10}$$

on the line $K = K_p$. Usually, the function $I(K_p)$ is analytic since it is obtained by the integration (4.8) with respect to a local variable. Thus, the internal energy has no singularity and the specific heat does not diverge on $K = K_p$. The phase diagram has the topology of figure 3(*a*); the F boundary is vertical and intersects with the OP boundary and the line $K = K_p$ at the same point, a multicritical point.

In the following subsections, we discuss several examples of gauge-symmetric models in the above class.

4.1. ±J Ising model

The non-random Ising model with pair interactions is written as

$$\mathcal{H}_0\{\phi\} = -J \sum_{(ij)\in\Lambda_B} \phi_i \phi_j \tag{4.11}$$

where the function ε was chosen to be

$$\varepsilon(\phi) = \phi \tag{4.12}$$

with $\phi \in \{-1, 1\} = \Phi$. The operation $\phi \circ \psi$ is regarded as the product $\phi\psi$ with $\bar{\phi} = \phi$ and $\phi_E = 1$, and the integration of the phase space is given by

$$\int \mathrm{d}\mu(\phi_i) \cdots = \sum_{\phi_i = \pm 1} \cdots. \tag{4.13}$$

The Hamiltonian and the probability weight of the associated random system are expressed as

$$\mathcal{H}\{\phi\}\{\omega\} = -J \sum_{(ij)\in\Lambda_B} \omega_{ij}\phi_i\phi_j \tag{4.14}$$

$$P(K_p, \{\omega\}) = \prod_{(ij)\in\Lambda_B} \tfrac{1}{2}(1 + (2p-1)\omega_{ij}) \tag{4.15}$$

where $\omega_{ij} \in \{-1, 1\} = \Phi$. This is called the $\pm J$ Ising model. If we define p by

$$\exp(-2K_p) = (1-p)/p \tag{4.16}$$

the $\pm J$ Ising model satisfies equations (4.1)–(4.5). Then, from equations (4.8)–(4.10), we obtain

$$I(K_p) = 2\cosh K_p \tag{4.17a}$$

$$E(K_p, K_p) = -N_B J \tanh K_p \tag{4.17b}$$

$$k_B T^2 C(K_p, K_p) \leqslant N_B J^2 \operatorname{sech}^2 K_p \tag{4.17c}$$

on the line $K = K_p$ which is called the Nishimori line [10].

The correlation functions in equation (2.34) are defined through the trivial representation

$$\gamma(\phi) = \phi. \tag{4.18}$$

In the case of the nearest-neighbour interactions on hypercubic lattices, paramagnetic, ferromagnetic and spin glass phases are likely to exist when $d \geqslant 2$ [1–8]. The phase diagram has the same topology as in figure 3(*a*); the F boundary is vertical and intersects with the OP boundary and the line $K = K_p$ at the multicritical point.

At $K_p = 0$ (or $p = 1/2$ from equation (4.16)), the existence of the spin glass phase at finite temperatures has been confirmed in three dimensions but has been refuted in two dimensions by several methods [1–3]. The phase diagrams in the region $K_p > 0$ for two and three dimensions have been obtained by numerical methods [4–8] and the results are consistent with figure 3(*a*). We remark that the spin glass phase of the $\pm J$ Ising model in two dimensions is sometimes called the random antiphase state [5–7, 43, 44]. Numerical evidence is accumulating for its existence [5–7].

4.2. XY gauge glass

Let us first consider the non-random classical XY (plane rotator) model;

$$\mathcal{H}_0\{\phi\} = -J \sum_{(ij)\in\Lambda_B} \cos(\phi_i - \phi_j). \tag{4.19}$$

where $\Phi = [0, 2\pi)$, with $\mathrm{d}\mu(\phi_i) = \mathrm{d}\phi_i$. The set Φ forms a group isomorphic to $O(2)$ if one defines the product operation by

$$\phi \circ \psi = \phi + \psi \pmod{2\pi} \tag{4.20}$$

with $\phi_E = 0$ and $\bar{\phi} = -\phi$.

Following the general strategy in section 2, we derive the gauge-symmetric random model associated with the non-random model (4.19) by introducing quenched gauge variables. The Hamiltonian and the probability weight of the gauge-symmetric model, equations (4.4) and (4.5), are expressed as

$$\mathcal{H}\{\phi\}\{\omega\} = -J \sum_{\langle ij\rangle \in \Lambda_B} \cos(\phi_i - \phi_j + \omega_{ij}) \tag{4.21}$$

and

$$P\big(K_p, \{\omega\}\big) = \prod_{\langle ij\rangle \in \Lambda_B} I(K_p)^{-1} \exp\big(K_p \cos(\omega_{ij})\big) \tag{4.22}$$

where $\omega_{ij} \in [0, 2\pi)$. Then, from equations (4.8)–(4.10), we obtain

$$I(K_p) = 2\pi I_0(K_p) \tag{4.23a}$$

$$E(K_p, K_p) = -N_B J \frac{I_1(K_p)}{I_0(K_p)} \tag{4.23b}$$

$$k_B T^2 C(K_p, K_p) \leqslant N_B J^2 \Big(1 - \frac{I_1}{K_p I_0} - \Big(\frac{I_1}{I_0}\Big)^2\Big) \tag{4.23c}$$

on $K = K_p$, where $I_m(K_p)$ are the modified Bessel functions.

The correlation functions in equation (2.34) are defined through the representation

$$\gamma(\phi) = \mathrm{e}^{\mathrm{i}\phi}. \tag{4.24}$$

In the case of the nearest-neighbour interactions on hypercubic lattices, paramagnetic, ferromagnetic and gauge glass phases are expected to appear when $d \geqslant 3$ [24–29, 45]. The phase diagram must have the same topology as in figure 3(*a*); the F boundary is vertical and intersects the OP boundary and the line $K = K_p$ at the same point, a multicritical point. When $d = 2$, the KT phase exists in the low-temperature region of the non-random system [41, 42]. If the KT phase extends into the finite K_p region with the same properties proposed for the ferromagnetic phase in section 3.2, the phase diagram has a topology similar to that in figure 4; the KT boundary is vertical and intersects the OP boundary and the line $K = K_p$ at the same point. The gauge glass phase in the region $K_p < K_c(0)$ may not exist in two dimensions [24, 25].

Note that the $\pm J$-type random model

$$\mathcal{H} = -\sum_{\langle ij\rangle \in \Lambda_B} J_{ij} \cos(\phi_i - \phi_j) \tag{4.25}$$

is not gauge symmetric in the case of the XY model. It is believed that the lower critical dimension of the $\pm J$ XY model is four or greater, although the possibility of a chirality glass transition has been pointed out for the $\pm J$ XY model in three dimensions [13–18].

Recently, the XY gauge glass of equation (4.21) has attracted much attention [19–46]. A real-space renormalization group method was applied to find a re-entrant transition in two and $2+\epsilon$ dimensions [19–21]. Experimental as well as Monte Carlo studies indicate no re-entrance [22, 23, 47, 48] in two dimensions. A Migdal–Kadanoff-type renormalization group calculation in three dimensions also failed to discover re-entrance [29]. Our argument in the

present paper shows that the XY gauge glass has a vertical F boundary (or KT boundary), which means the absence of re-entrance. A few remarks are in order on this point.

First, the present argument assumes the existence of a finite region of ferromagnetic (or KT) phase. If this ordered phase is unstable against weak gauge randomness as suggested in [46], the phase diagram will be filled simply with a paramagnetic state except at the non-random limit.

A second remark is on the probability distribution function (4.22). In most of the theoretical studies referred to above, the probability distribution of the gauge variable ω_{ij} is Gaussian. In particular, the renormalization group calculations [19–21] use the Gaussian ω_{ij} and the Villain potential (a periodic Gaussian interaction) [49] instead of equations (4.21) and (4.22). Our argument also applies to such a case. If the pair interaction is the Villain potential

$$V(\phi_i - \phi_j + \omega_{ij}) = \log \sum_{n=-\infty}^{\infty} \exp\{-\tfrac{1}{2}K(\phi_i - \phi_j + \omega_{ij} - 2\pi n)^2\} \tag{4.26}$$

with $\phi_i, \phi_j \in [0, 2\pi)$ and $\omega_{ij} \in \boldsymbol{R}$, the Gaussian probability distribution automatically acquires periodicity

$$P(K_p, \omega_{ij}) \propto \sum_{m=-\infty}^{\infty} \exp\{-\tfrac{1}{2}K_p(\omega_{ij} - 2\pi m)^2\} \tag{4.27}$$

because the interaction (4.26) is not affected by the change of ω_{ij} by $2\pi m$. Thus, one may regard $\phi_i, \phi_j, \omega_{ij} \in [0, 2\pi)$. The interaction (4.26) and the probability weight (4.27) have the same form, which makes it possible to apply the general argument in section 2. We therefore conclude that the F boundary (or KT boundary) (if any) is vertical in this Villain model with Gaussian randomness.

Reger and Young [50] argued that the existence of transverse degrees of freedom may lead to re-entrance which is absent in the Ising case. The present argument shows that the mere existence of a transverse component is insufficient to induce re-entrance.

4.3. Z_q gauge glass

Let the set Φ be

$$\Phi_q = \left\{ \frac{2\pi m}{q} \middle| m = 0, 1, \ldots, q-1 \right\}. \tag{4.28}$$

With the same operation as equation (4.20), Φ_q forms the group Z_q, a subgroup of $O(2)$. The integration of the phase space becomes a summation as

$$\int \mathrm{d}\mu(\phi_i) \cdots = \sum_{\phi_i \in \Phi_q} \cdots. \tag{4.29}$$

If the Hamiltonian is chosen to be the same as in equation (4.19) with $\phi_i \in \Phi_q$, it is usually called the q-state clock model. It reduces to the Ising model and the XY model when $q = 2$ and $q \to \infty$, respectively. The Hamiltonian and the probability weight of a gauge-symmetric random model are expressed by equations (4.21) and (4.22) with $\omega_{ij} \in \Phi_q$ and

$$I(K_p) = \sum_{m=0}^{q-1} \exp\left(K_p \cos(2\pi m/q)\right). \tag{4.30}$$

The energy and a bound on the specific heat on $K = K_p$ are calculated from equations (4.9), (4.10) and (4.30). The correlation functions (2.34) are defined through equation (4.24). In the case of the nearest-neighbour interactions on hypercubic lattices, paramagnetic, ferromagnetic and gauge glass phases appear for $q \geqslant 2$ when $d \geqslant 3$ [1–3, 24–29, 51]. The phase diagram should have the same topology as in figure 3(a). The situation may be similar in two dimensions for $4 \geqslant q \geqslant 2$ since the non-random system has a ferromagnetic phase in this case and the Ising ($q = 2$) model has a ferromagnetic and spin glass phases (often called the random antiphase state) [4–7]. When $q \geqslant 5$ in two dimensions, both KT and ferromagnetic phases exist in the low-temperature region of the non-random system [52–54]. The phase diagram has the same topology as in figure 5.

The non-random q-state Potts model has the Hamiltonian

$$\mathcal{H}_0\{\phi\} = -J \sum_{\langle ij\rangle \in \Lambda_B} \delta[\phi_i \circ \bar{\phi}_j, 0] \tag{4.31}$$

where $\delta[\cdot, \cdot]$ is Kronecker's delta. The Hamiltonian and the probability weight of the gauge-symmetric model [30], equations (4.4) and (4.5), are expressed as

$$\mathcal{H}\{\phi\}\{\omega\} = -J \sum_{\langle ij\rangle \in \Lambda_B} \delta[\phi_i \circ \bar{\phi}_j \circ \omega_{ij}, 0] \tag{4.32}$$

$$P\big(K_p, \{\omega\}\big) = \prod_{\langle ij\rangle \in \Lambda_B} I(K_p)^{-1} \exp\big(K_p \delta[\omega_{ij}, 0]\big) \tag{4.33}$$

where $\omega_{ij} \in \Phi_q$. Then, from equations (4.8)–(4.10), we obtain

$$I(K_p) = \mathrm{e}^{K_p} + q - 1 \tag{4.34a}$$

$$E(K_p, K_p) = -N_B J \frac{\mathrm{e}^{K_p}}{\mathrm{e}^{K_p} + q - 1} \tag{4.34b}$$

$$k_B T^2 C(K_p, K_p) \leqslant N_B J^2 \frac{\mathrm{e}^{K_p}(q-1)}{\{\mathrm{e}^{K_p} + q - 1\}^2}. \tag{4.34c}$$

In the case of the nearest-neighbour interactions on hypercubic lattices, paramagnetic, ferromagnetic and gauge glass phases may appear for $q \geqslant 2$ when $d \geqslant 3$ [51]. The correlation functions (2.34) are defined through equation (4.24). The phase diagram has the same topology as in figure 3(a).

Note that the Hamiltonian (4.32) is invariant under a wider class of symmetry operation belonging to the 'symmetric group' of degree q. The group Z_q is a subgroup of the symmetric group of degree q. Since the number of elements of the symmetric group of degree q is much larger than q, we cannot define the transformation $U_\psi^{(i)}$ for that group. Although it is possible to define a gauge-symmetric random model in which each spin variable $\phi_i \in \Phi_q$ and each random variable ω_{ij} have a symmetry due to the symmetric group of degree q, the resulting model is not in the present class of gauge-symmetric random models.

4.4. SU(2) gauge glass

The present theory can be applied to non-Abelian groups Φ. For example, the non-random Hamiltonian of the nearest-neighbour $SU(2)$ model is written by two-dimensional unitary matrices $\phi_i \in SU(2)$ [55];

$$\mathcal{H}_0\{\phi\} = -J \sum_{(ij)\in\Lambda_B} \mathrm{Tr}\,\phi_i \phi_j^\dagger. \tag{4.35}$$

Let us introduce a parametrization of a 2×2 $SU(2)$ matrix by the Pauli matrices $\tau = (\tau_1, \tau_2, \tau_3)$,

$$\phi_i = \sigma_i + \mathrm{i}\tau \cdot (\phi_i)_i \tag{4.36}$$

where $(\sigma_i, \pi_{i1}, \pi_{i2}, \pi_{i3})$ are four real variables on the ith site with

$$\sigma_i^2 + (\phi_i)_i \cdot (\phi_i)_i = 1. \tag{4.37}$$

Then the above Hamiltonian is rewritten as

$$\mathcal{H}_0\{\phi\} = -2J \sum_{(ij)\in\Lambda_B} (\sigma_i\sigma_j + (\phi_i)_i \cdot (\phi_i)_j) \tag{4.38}$$

which is equivalent to the $O(4)$ model. The invariant measure is expressed as

$$\mathrm{d}\mu(\phi_i) = \delta(\sigma_i^2 + (\phi_i)_i \cdot (\phi_i)_i - 1)\mathrm{d}\sigma_i\, \mathrm{d}(\phi_i)_i \tag{4.39}$$

where $\delta(\cdot)$ is the delta function.

The Hamiltonian and the probability weight of the gauge-symmetric model, equations (4.4) and (4.5), are expressed as

$$\mathcal{H}\{\phi\}\{\omega\} = -J \sum_{(ij)\in\Lambda_B} \mathrm{Tr}\,\phi_i \omega_{ij} \phi_j^\dagger \tag{4.40}$$

and

$$P(K_p, \{\omega\}) = \prod_{(ij)\in\Lambda_B} I(K_p)^{-1} \exp\left(K_p \,\mathrm{Tr}\,\omega_{ij}\right) \tag{4.41}$$

where $\omega_{ij} \in SU(2)$. Let $\omega_{ij} = x_{ij} + \mathrm{i}\tau \cdot y_{ij}$. Then the Hamiltonian (4.40) is written as

$$\mathcal{H}\{\phi\}\{\omega\} = -2J \sum_{(ij)\in\Lambda_B} (\sigma_i\sigma_j + (\phi_i)_i \cdot (\phi_i)_j)x_{ij} + (\sigma_i(\phi_i)_j - \sigma_j(\phi_i)_i) \cdot y_{ij}. \tag{4.42}$$

From equations (4.8)–(4.10), we obtain

$$I(K_p) = \frac{2\pi^2}{K_p} I_1(2K_p) \tag{4.43a}$$

$$E(K_p, K_p) = -2N_B J \frac{I_2(2K_p)}{I_1(2K_p)} \tag{4.43a}$$

$$k_B T^2 C(K_p, K_p) \leqslant 4N_B J^2 \left(1 - \frac{3I_2}{2K_p I_1} - \left(\frac{I_2}{I_1}\right)^2\right) \tag{4.43a}$$

on $K = K_p$, where $I_m(K_p)$ are the modified Bessel functions.

In the case of the nearest-neighbour interactions on hypercubic lattices, paramagnetic, ferromagnetic and gauge glass phases may appear for sufficiently large d. The correlation functions in equation (2.34) are defined through the representation of equation (4.18); in this case, ϕ is a 2×2 matrix. Then, from equation (4.36), f_γ forms the ordinary ferromagnetic correlation in the $O(4)$ model

$$\mathrm{Tr}\,\phi_0\phi_r = 2(\sigma_0\sigma_r + (\phi_i)_0 \cdot (\phi_i)_r). \tag{4.44}$$

All results in section 3 apply to this $SU(2)$ model.

4.5. SOS gauge glass

The present theory can be applied to non-compact groups Φ; an example is $\mathbb{Z}$ as an additive group. We need a slightly different formulation for non-compact groups because the normalization constant c diverges. The formulation for the non-compact case is presented in the appendix. As an example, the SOS (solid-on-solid) Hamiltonian [56] is expressed as

$$\mathcal{H}_0\{\phi\} = J \sum_{\langle ij\rangle \in \Lambda_B} |\phi_i - \phi_j|^t \tag{4.45}$$

where t is a positive constant, $\phi_i \in \mathbb{Z}$, and

$$\int d\mu(\phi_i) \cdots = \sum_{\phi_i=-\infty}^{\infty} \cdots . \tag{4.46}$$

The Hamiltonian and the probability weight of the gauge-symmetric model, equations (4.4) and (4.5), are expressed as

$$\mathcal{H}\{\phi\}\{\omega\} = J \sum_{\langle ij\rangle \in \Lambda_B} |\phi_i - \phi_j + \omega_{ij}|^t \tag{4.47}$$

$$P\big(K_p, \{\omega\}\big) = \prod_{\langle ij\rangle \in \Lambda_B} I(K_p)^{-1} \exp(-K_p|\omega_{ij}|^t) \tag{4.48}$$

with $\omega_{ij} \in \mathbb{Z}$. As shown in the appendix, many results in sections 2 and 3 can be applied to such non-compact cases. The normalization factor is

$$I(K_p) = \sum_{\omega=-\infty}^{\infty} \exp(-K_p|\omega|^t) \tag{4.49}$$

and the internal energy and a bound on the specific heat under the condition $K = K_p$ are derived as in equations (4.9) and (4.10).

We do not discuss the topology of phase diagram, since it is not clear whether or not relevant order parameters can be expressed as in equation (2.34).

5. Applications to pair interactions II: Gaussian randomness for Abelian group

In the present section, we restrict our attention to the case where Φ is compact and Abelian, such as $O(2)$ or Z_q. Let us choose $\Phi \subseteq [0, 2\pi)$ with the operation (4.20). Irreducible representations are one-dimensional and unitary, $\gamma(\phi) = e^{ik\phi}$.

We introduce a new class of gauge-symmetric models associated with the non-random Hamiltonian of equations (4.1)–(4.3). The function $\varepsilon(\phi)$ is expanded in general in the Fourier series as

$$\varepsilon(\phi) = \sum_n{}' a_n s_n \cos n\phi \tag{5.1}$$

where $a_n > 0$ and $s_n = \pm 1$. The summation of equation (5.1) runs only over n with $a_n \neq 0$.

Let us assign a random variable $\omega_{ij,n} \in \mathbb{C}$ for every pair $\langle ij\rangle$ and every n with $a_n \neq 0$. Thus $\Omega_{\text{tot}} = \mathbb{C}^{N_B N_a}$ (N_a is the number of non-zero a_n). The complex Gaussian distribution is given by

$$P\big(K_p, \{\omega\}\big) = \prod_{\langle ij\rangle \in \Lambda_B} \prod_n{}' \frac{a_n}{2\pi} \exp\big(-a_n|\omega_{ij,n} - s_n K_p|^2/2\big) \tag{5.2}$$

with the measure $d\nu\{\omega\} = \prod_{\langle ij\rangle\in\Lambda_B}\prod_n{}' d\nu(\omega_{ij,n})$; the region of integration $\int d\nu(\omega_{ij,n})$ is the whole complex plane, and

$$d\nu(\omega_{ij,n}) = d\,\mathrm{Re}\,\omega_{ij,n}\, d\,\mathrm{Im}\,\omega_{ij,n}. \tag{5.3}$$

The symbols Re(·) and Im(·) indicate the real and imaginary parts, respectively. In the usual notation, each variable is expressed by J and J_0:

$$K = \beta J \tag{5.4a}$$

$$K_p = J_0/J \tag{5.4b}$$

$$\omega_{ij,n} = J_{ij,n}/J. \tag{5.4c}$$

The gauge-symmetric Gaussian random Hamiltonian associated with equation (4.1) with equation (5.1) is given as

$$\mathcal{H}\{\phi\}\{\omega\} = -J\sum_{\langle ij\rangle\in\Lambda_B}\sum_n{}' a_n s_n \,\mathrm{Re}\left[\omega_{ij,n}e^{in(\phi_i-\phi_j)}\right]. \tag{5.5}$$

The gauge transformation $V_{\{\psi\}} = \prod_{\langle ij\rangle\in\Lambda_B}\prod_n{}' V_{\psi_i\circ\psi_j}^{(ij,n)}$ is constructed as the local transformation

$$V_\psi^{(ij,n)} : \omega_{ij,n} \to \omega'_{ij,n} = \omega_{ij,n}e^{in\psi} \tag{5.6}$$

by assuming $i > j$. Clearly, the measure $d\nu(\omega_{ij,n})$ is invariant with respect to $V_\psi^{(ij,n)}$ for any $\psi \in \Phi$. Note that K_p is assumed to be non-negative. From equation (5.6), the probability weight (5.2) is transformed as in equation (2.19) with

$$D\{\omega\} = \prod_{\langle ij\rangle\in\Lambda_B}\prod_n{}' \frac{a_n}{2\pi}\exp(-a_n|\omega_{ij,n}|^2/2) \tag{5.7}$$

and

$$Y(K_p)^{1/N_B} = I(K_p) = \prod_n{}' \exp(a_n K_p^2/2) = \exp\left(\tilde{\varepsilon}(0)K_p^2/2\right) \tag{5.8}$$

where

$$\tilde{\varepsilon}(\phi) = \sum_n{}' a_n\cos n\phi. \tag{5.9}$$

It is easy to see that the above system satisfies conditions I to V. *Consequently, if the* Hamiltonian is expressed by equation (4.1) with equations (4.2) and (4.3) and Φ is compact and Abelian, one can derive a gauge-symmetric random model of equations (5.2) and (5.5).

The internal energy and a bound on the specific heat for $K = K_p$ *are obtained from* equations (4.9) and (4.10):

$$E(K_p, K_p) = -N_B J\tilde{\varepsilon}(0)K_p = -N_B\tilde{\varepsilon}(0)J_0 \tag{5.10a}$$

$$k_B T^2 C(K_p, K_p) \leqslant N_B\tilde{\varepsilon}(0)J^2. \tag{5.10b}$$

Similarly to the class of models in section 4, the internal energy has no singularity, the specific heat does not diverge on $K = K_p$, and the phase diagram has the same topology as in figure 3(*a*).

In the case of the infinite-range model, $N_B = N(N-1)/2$. The parameters J and J_0 are scaled as $J \to J/\sqrt{N}$ and $J_0 \to J_0/N$ so that the free energy is extensive. From equation (5.4), this scaling is equivalent to $K \to K/\sqrt{N}$ and $K_p \to K_p/\sqrt{N}$. The parameter of the modified model is also needed to be scaled as $a \to a/\sqrt{N}$. Then the same arguments as before can be developed for the topology of phase diagram using these scaled parameters.

5.1. Gaussian random Ising model

The best studied model in this class is the Ising system, for example the infinite-range Sherrington–Kirkpatrick (SK) model, and its finite-dimensional version [57–59]. Let the interaction be

$$\varepsilon(\phi) = \cos\phi \tag{5.11}$$

with $\phi_i \in \{0, \pi\} = \Phi_2$ and $S_i = \exp(i\phi_i)$. Then we have

$$\mathcal{H} = -J \sum_{\langle ij\rangle \in \Lambda_B} \mathrm{Re}(\omega_{ij}) S_i S_j \qquad (S_i = \pm 1) \tag{5.12}$$

and

$$P(K_p, \{\omega\}) = \prod_{\langle ij\rangle \in \Lambda_B} \frac{1}{2\pi} \exp\left[-(\mathrm{Re}\,\omega_{ij} - K_p)^2/2 - (\mathrm{Im}\,\omega_{ij})^2/2 \right] \tag{5.13}$$

where $\omega_{ij} \in \mathbb{C}$. Clearly, the imaginary part $\mathrm{Im}\,\omega_{ij}$ plays no role in this case.

The energy and a bound on the specific heat on $K = K_p$ are obtained as in equation (5.10) with $\tilde{\varepsilon}(0) = 1$. The phase diagram has the same topology as in figure 3(*a*). The internal energy of the replica symmetric solution of the SK model [59] is actually equal to equation (5.10*a*) if $K = K_p$ or $\beta J^2 = J_0$. The specific heat for $K = K_p$ derived from the replica symmetric solution of the SK model is given by the right-hand side of equation (5.10*b*) in the paramagnetic phase and satisfies the inequality (5.10*b*) in the ferromagnetic phase. The boundary between the ferromagnetic (mixed) phase and the spin glass phase is vertical [33] as is required from our general considerations.

5.2. Other Gaussian random models

Models in the same class are also defined for other Abelian groups. The gauge-symmetric Gaussian random XY model derived from equation (4.19) is given by

$$\mathcal{H}\{\phi\}\{\omega\} = -J \sum_{\langle ij\rangle \in \Lambda_B} \mathrm{Re}\left[\omega_{ij} e^{i(\phi_i - \phi_j)}\right] \tag{5.14}$$

and the probability weight is equation (5.13) with $\omega_{ij} \in \mathbb{C}$. The energy and a bound on the specific heat on $K = K_p$ are in equation (5.10) with $\tilde{\varepsilon}(0) = 1$.

The gauge-symmetric Gaussian random q-state clock model derived from equation (4.19) is also defined by equation (5.14) with $\phi \in \Phi_q$ (see equation (4.28)) and the probability weight of equation (5.13) with $\omega_{ij} \in \mathbb{C}$. The energy and the specific heat on $K = K_p$ are obtained as in equation (5.10) with $\tilde{\varepsilon}(0) = 1$.

The gauge-symmetric Gaussian random q-state Potts model [30] derived from equation (4.19) is also defined by

$$\mathcal{H}\{\phi\}\{\omega\} = -\frac{J}{q} \sum_{\langle ij\rangle \in \Lambda_B} \sum_{n=0}^{q-1} \mathrm{Re}\left[\omega_{ij,n} e^{in(\phi_i - \phi_j)}\right] \tag{5.15}$$

with $\phi \in \Phi_q$ and $\omega_{ij,n} \in \mathbb{C}$. The energy and a bound on the specific heat for $K = K_p$ are given as in equation (5.10) with $\tilde{\varepsilon}(0) = 1$.

In each model, the phase diagram has a vertical F boundary which intersects the OP boundary and the line $K = K_p$ at the multicritical point.

6. Many-body interactions

The random energy model [60] is obtained from the infinite-range Ising system with p-body interactions in the limit $p \to \infty$. The $p = 2$ case corresponds to the SK model. The present theory can be applied to many-body interactions if the set Φ is Abelian.

For simplicity of presentation, we restrict our attention to the case of four-body interactions on every plaquette on a d-dimensional hypercubic lattice. Let $\Lambda_P \subset \Lambda^4$ be a set of N_P plaquettes on Λ. The non-random Hamiltonian is assumed to be

$$\mathcal{H}_0\{\phi\} = -J \sum_{\langle ijkl\rangle \in \Lambda_P} \varepsilon(\phi_i \circ \bar{\phi}_j \circ \phi_k \circ \bar{\phi}_l) \tag{6.1}$$

with equations (4.2) and (4.3), where a plaquette $\langle ijkl\rangle$ is taken only once in the summation.

First, we introduce a class similar to the one discussed in section 4. We assign a random variable $\omega_{ijkl} \in \Phi$ to each plaquette, with which the measure $d\mu(\omega_{ijkl})$ is chosen to be the same as that of the spin degrees of freedom. Thus, $\Omega_{tot} = \Phi^{N_P}$, and $d\nu\{\omega\} = \prod_{\langle ij\rangle \in \Lambda_B} d\mu(\omega_{ijkl})$. The Hamiltonian is set to be

$$\mathcal{H}\{\phi\}\{\omega\} = -J \sum_{\langle ijkl\rangle \in \Lambda_P} \varepsilon(\omega_{ijkl} \circ \phi_i \circ \bar{\phi}_j \circ \phi_k \circ \bar{\phi}_l) \tag{6.2}$$

with the probability weight

$$P\big(K_p, \{\omega\}\big) = \prod_{\langle ijkl\rangle \in \Lambda_P} I(K_p)^{-1} \exp\big(K_p \varepsilon(\omega_{ijkl})\big). \tag{6.3}$$

The gauge transformation

$$V_{\{\psi\}} = \prod_{\langle ijkl\rangle \in \Lambda_P} V^{(ijkl)}_{\psi_i \circ \bar{\psi}_j \circ \psi_k \circ \bar{\psi}_l} \tag{6.4}$$

is constructed from the local transformation defined by

$$V^{(ijkl)}_{\psi} : \omega_{ijkl} \to \omega'_{ijkl} = \omega_{ijkl} \circ \psi \tag{6.5}$$

where we assume $i > j > k > l$. Note that $V^{(ijkl)}_{\psi}$ forms a group isomorphic to Φ, and $V_{\{\psi\}}$ is homomorphic to Φ^N. Thus, it is easy to see that the above system satisfies conditions I to V. Consequently, if the Hamiltonian is expressed by equation (6.1) with equations (4.2) and (4.3) and Φ is Abelian, one can derive a gauge-symmetric random model in equations (6.2) and (6.3).

The normalization factor $Y(K_p)$ is obtained by equations (4.7) and (4.8) with N_B replaced by N_P. Using the general result in equations (2.30*b*) and (2.31*b*), we find that the internal energy and a bound on the specific heat under the condition $K = K_p$ are expressed by equations (4.9) and (4.10) with N_B replaced by N_P. Thus the internal energy has no singularity, the specific heat does not diverge if $K = K_p$, and the phase diagram has the same topology as in figure 3(*a*).

It is possible to introduce the class of Gaussian randomness. If the function $\varepsilon(\phi)$ is expressed by equation (5.1), the gauge-symmetric Gaussian random Hamiltonian is defined by

$$\mathcal{H}\{\phi\}\{\omega\} = -J \sum_{\langle ijkl\rangle \in \Lambda_P} {\sum_n}' a_n s_n \operatorname{Re}\left[\omega_{ijkl,n} e^{in(\phi_i - \phi_j + \phi_k - \phi_l)}\right]. \tag{6.6}$$

Each random variable $\omega_{ijkl,n} \in \mathbb{C}$ obeys the complex Gaussian distribution like equation (5.13). If the gauge transformation (6.4) is constructed by the local transformation defined by

$$V_{\psi}^{(ijkl,n)} : \omega_{ijkl,n} \to \omega'_{ijkl,n} = \omega_{ijkl,n} e^{in\psi} \tag{6.7}$$

the above system satisfies conditions I to V. All results in sections 2 and 3 apply. In particular the random energy model with a non-symmetric Gaussian probability should satisfy these results.

7. Summary

We have applied the method of gauge transformation to random spin systems with various kinds of symmetry. The conditions on the Hamiltonian and the probability weight of random configurations were derived so that the present theory is applicable. All gauge-symmetric models share significant common properties; the internal energy and an upper bound on the specific heat have simple forms on a special line in the K_p–K plane, where K is the effective coupling and K_p controls the randomness. In the cases discussed in sections 5–7, it has been shown that the internal energy has no singularity and the specific heat does not diverge. If the low-temperature phase of the non-random system extends into the weakly random case ($K_p \gg 1$), the phase boundary in the K_p–K plane is vertical to the K_p axis, implying the lack of re-entrance (see figures 3(a), 4 and 5). We note that the conclusion on the vertical boundary has not been derived completely rigorously because we assumed the existence of a ferromagnetic phase in the modified model, a plausible but unproved property. The results on the internal energy and specific heat are free from any unproved assumptions. We have introduced several kinds of gauge-symmetric random models with symmetry groups including $O(2)$, Z_q and $SU(2)$ as examples.

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Appendix. Formulation without the normalization parameter c

In the case with diverging c, we cannot use the equations involving c, such as equations (2.24) and (2.27). However, if the left-hand sides of these equations have finite values, one may modify these equations and derive many of the results which do not involve c. We can prove, for instance, equation (2.30b) without using c as follows. From the first line of equation (2.30a),

$$E(K, K_p) = -\int \mathrm{d}\nu\{\omega\} P(K_p, \{\omega\}) Z(K, \{\omega\})^{-1} \frac{\partial}{\partial\beta} Z(K, \{\omega\})$$

$$= -\int \mathrm{d}\mu\{\phi\} \int \mathrm{d}\nu\{\omega\} V_{\{\phi\}} \frac{P(K_p, \{\omega\})}{Z(K, \{\omega\})} \frac{\partial}{\partial\beta} \exp\left(-K\tilde{\mathcal{H}}\{\phi\}\{\omega\}\right)$$

$$= -Y(K_p)^{-1}\int d\mu\{\phi\}\int d\nu\{\omega\}D\{\omega\}\frac{\exp\left(-K_p\tilde{\mathcal{H}}\{\bar{\phi}\}\{\omega\}\right)}{Z(K,\{\omega\})}$$

$$\times\frac{\partial}{\partial\beta}\exp\left(-K\tilde{\mathcal{H}}\{\phi_E\}\{\omega\}\right)$$

$$= -Y(K_p)^{-1}\int d\nu\{\omega\}D\{\omega\}\frac{Z(K_p,\{\omega\})}{Z(K,\{\omega\})}\frac{\partial}{\partial\beta}\exp\left(-K\tilde{\mathcal{H}}\{\phi_E\}\{\omega\}\right) \tag{A.1}$$

If $K = K_p$, the two Zs cancel out to yield

$$E(K_p,K_p) = -JY(K_p)^{-1}\frac{\partial}{\partial K_p}\int d\nu\{\omega\}D\{\omega\}\exp\left(-K_p\tilde{\mathcal{H}}\{\phi_E\}\{\omega\}\right)$$

$$= -J\frac{\partial}{\partial K_p}\ln Y(K_p) \tag{A.2}$$

where we have used the invariance of equations (2.8*c*) and (2.16), and the properties expressed in equations (2.18*a*) and (2.19). In general, one can prove for an arbitrary function $R\{\phi\}\{\omega\}$ of $\{\phi\}$ and $\{\omega\}$ that

$$\left[\langle R\{\phi\}\{\omega\}\rangle_K\right]_{K_p} = \frac{Y(K)}{Y(K_p)}\left[\frac{Z(K_p,\{\omega\})}{Z(K,\{\omega\})}\langle V_{\{\phi\}}R\{\bar{\phi}\}\{\omega\}\rangle_{K_p}\right]_K. \tag{A.3}$$

From equation (A.3) all results in section 2, such as equations (2.31*b*), (2.38), (2.39), (2.45), (2.48)–(2.54), can be rederived without using c. For instance, using equation (A.3), we find that

$$\left[\langle R\{\phi\}\{\omega\}\rangle_{K_p+a}\right]_{K_p} = \left\{\langle V_{\{\phi\}}R\{\bar{\phi}\}\{\omega\}\rangle_{K_p}\right\}^a_{K_p}. \tag{A.4}$$

If $R\{\phi\}\{\omega\}$ is independent of $\{\phi\}$ (to be written as $Q\{\omega\}$) and gauge invariant, we find from equation (A.4)

$$\left[Q\{\omega\}\right]_{K_p} = \left\{Q\{\omega\}\right\}^a_{K_p} \tag{A.5}$$

which is equivalent to equation (2.45).

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